Differential Geometry

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1. Calculus on Euclidean Space

1.1. Directional Direvatives

The directional derivative of a function $f(\mathbf{p})$, with respect to a tangent vector $\mathbf{v}$ is a real number

$$\begin{aligned}\mathbf{v}[f] &\equiv \left.\frac{d}{dt}\right|_{t=0} f(\mathbf{p}+t\mathbf{v}) \\ &= \sum \mathbf{v}_i U_i(\mathbf{p})[f] \\ &= \sum \mathbf{v}_i \frac{\partial f}{\partial x_i}(\mathbf{p}) \end{aligned} \tag{1}$$

The differential df of f is the 1-form such that $df(v_p) = v_p[f]$ for all tangent vector v_p [2].

Lemma. *Let α be a curve in $\mathbb{R}^3$ and let f be a differentiable function on $\mathbb{R}^3$. Then*

$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t) \tag{2}$$

Proof. Since $\alpha' = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt}\right)$, then by definiton of directional derivative,

$$\alpha'(t)[f] = \sum_i \frac{d\alpha_i}{dt}(t)\frac{\partial f}{\partial x_i}(\alpha(t)) = \frac{d(f(\alpha))}{dt}(t) \qquad \square$$

Two useful identities are

$$U_i[f] = \frac{\partial f}{\partial x_i} \tag{3}$$

$$dx_i(v) = v_i \tag{4}$$

Differential forms on $\mathbb{R}^3$ have the following 1-1 correspondences: 0-forms can be identified with scalar functions; 1-forms can be identified with vector fields; 2-forms can also be identified with vector fields via right-hand rule; 3-forms can be identified with scalar functions.

$$\sum_i f_i dx_i \overset{(1)}{\longleftrightarrow} \sum_i f_i U_i \overset{(2)}{\longleftrightarrow} f_1 dx_2 dx_3 + f_2 dx_3 dx_1 + f_3 dx_1 dx_2$$

Therefore we have

$$\begin{aligned} f &= f(x_1, \ldots, x_i, \ldots) \\ df &\overset{(1)}{\longleftrightarrow} \operatorname{grad} f \\ &= \sum_i \frac{\partial f}{\partial x_i} U_i \\ V &= \sum_i f_i U_i \end{aligned} \tag{5}$$

$$\begin{aligned} \text{If an 1 form } \phi \overset{(1)}{\longleftrightarrow} V\text{, then } d\phi &\overset{(2)}{\longleftrightarrow} \operatorname{curl} V \\ &= \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)U_1 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)U_2 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)U_3 \end{aligned} \tag{6}$$

$$\begin{aligned} \text{If a 2 form } \eta \overset{(1)}{\longleftrightarrow} V\text{, then } d\eta &= (\operatorname{div} V) dx dy dz \\ &= \left(\sum_i \frac{\partial f_i}{\partial x_i}\right) dx dy dz \end{aligned} \tag{7}$$

The correspondences can be summarized as follows.

0-form		$f \mapsto \nabla f$	gradient
1-form	$\omega \mapsto d\omega$	$V \mapsto \nabla \times V$	curl
2-form		$V \mapsto \nabla \cdot V$	divergence
0-form	$d(d\omega) = 0$	$\nabla \times (\nabla f) = 0$	
1-form		$\nabla \cdot (\nabla \times V) = 0$	

Table 1. Correspondences between forms and functions, vector fields

In summary, on an open subset U of $\mathbb{R}^3$, there are identifications

$$\begin{array}{ccccccc}
\Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \Omega^3(U) \\
\simeq\updownarrow & & \simeq\updownarrow & & \simeq\updownarrow & & \simeq\updownarrow \\
C^\infty(U) & \xrightarrow[\text{grad}]{} & \mathfrak{X}(U) & \xrightarrow[\text{curl}]{} & \mathfrak{X}(U) & \xrightarrow[\text{div}]{} & C^\infty(U)
\end{array}$$

1.2. Vector Fields

A vector field $\boldsymbol{W}$ on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector $\boldsymbol{W}_p$ in $T_p(\mathbb{R}^n)$ [4].

$$\boldsymbol{W}_p = \sum w^i(\mathbf{p}) \frac{\partial}{\partial x^i}\bigg|_p \tag{8}$$

where w^i are real-valued functions on U. Define a new function $\boldsymbol{W}f$ on U by

$$(\boldsymbol{W}f)(\mathbf{p}) = \boldsymbol{W}_p f = \sum w^i(\mathbf{p}) \frac{\partial f}{\partial x^i}\bigg|_p \tag{9}$$

or simply

$$\boldsymbol{W}f = \sum w^i \frac{\partial f}{\partial x^i} \tag{10}$$

1.3. Covariant Direvatives

The covariant derivative of a vector field $\boldsymbol{W} = \sum \boldsymbol{w_i U_i}$ with respect to a tangent vector $\mathbf{v}$ is the tangent vector

$$\begin{aligned}
\nabla_v \boldsymbol{W} &\equiv \frac{d}{dt}\bigg|_{t=0} \boldsymbol{W}(\mathbf{p} + t\mathbf{v}) \\
&= \sum \mathbf{v}[w_i] U_i(\mathbf{p}) \\
&= \begin{pmatrix} \mathbf{v}[w_1] \\ \mathbf{v}[w_2] \\ \mathbf{v}[w_3] \end{pmatrix}
\end{aligned} \tag{11}$$

Example. suppose $\boldsymbol{W} = x^2 U_1 + yzU_3$, and $\mathbf{v} = (-1, 0, 2)$ at $\mathbf{p} = (2, 1, 0)$, then

$$\begin{aligned}
\mathbf{p} + t\mathbf{v} &= (2 - t, 1, 2t) \\
\boldsymbol{W}(\mathbf{p} + t\mathbf{v}) &= (2 - t)^2 U_1 + 2\text{tU}_3 \\
\nabla_v \boldsymbol{W} &= \boldsymbol{W}(\mathbf{p} + t\mathbf{v})'(0) \\
&= -4U_1 + 2U_3
\end{aligned}$$

The covariant derivative of a vector field $\boldsymbol{W}$ with respect to a vector field $\boldsymbol{V}$ is the vector field

$$\nabla_V \boldsymbol{W} = \sum \boldsymbol{V}[w_i] U_i \tag{12}$$

Example. suppose $\boldsymbol{W} = x^2 U_1 + yzU_3$, and $\boldsymbol{V} = (y - x)U_1 + xyU_3$, then

$$\begin{aligned}
\boldsymbol{V}[x^2] &= (y - x)U_1[x^2] \\
&= 2x(y - x) \\
\boldsymbol{V}[yz] &= xyU_3[yz] \\
&= xy^2 \\
\nabla_V \boldsymbol{W} &= 2x(y - x)U_1 + xy^2 U_3
\end{aligned}$$

THEOREM. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be tangent vectors to* $\mathbb{R}^3$ *at* $\mathbf{p}$, *and let* $\boldsymbol{Y}$ *and* $\boldsymbol{Z}$ *be vector fields on* $\mathbb{R}^3$. *Then for numbers* a, b *and functions* f,

$$\begin{aligned}
\nabla_{av+bw} Y &= a\nabla_v Y + b\nabla_w Y \\
\nabla_v (aY + bZ) &= a\nabla_v Y + b\nabla_v Z \\
\nabla_v (fY) &= v[f]Y(\mathbf{p}) + f(\mathbf{p})\nabla_v Y \\
v[Y \cdot Z] &= \nabla_v Y \cdot Z(\mathbf{p}) + Y(\mathbf{p}) \cdot \nabla_v Z
\end{aligned}$$

1.4. Differential Forms

DEFINITION. *The alternating multilinear functions with k arguments on a vector space are called multicovectors of degree k, or $k-$* covectors *for short.*

DEFINITION. *A 1-form ϕ on $\mathbb{R}^3$ is a real-valued function on the set of all tangent vectors to $\mathbb{R}^3$ such that ϕ is linear at each point, that is,*

$$\phi(a\mathbf{v}+b\mathbf{w})=a\phi(\mathbf{v})+b\phi(\mathbf{w})$$

for any numbers a,b and tangent vectors $\mathbf{v},\mathbf{w}$ at the same point of $\mathbb{R}^3$.

Let f and g be real-valued functions on $\mathbb{R}^2$. It's proved that

$$df\wedge dg = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} dxdy \tag{13}$$

Therefore we have

$$\begin{aligned} dy\wedge dx &= \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \end{vmatrix} dxdy \\ dydx &= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} dxdy \\ dydx &= -dxdy \qquad (14) \end{aligned}$$

THEOREM. *Let f and g be functions (0-forms), ϕ and ψ are 1-forms. Then*

$$\begin{aligned} d(fg) &= (df)g+f(dg) \\ d(f\phi) &= df\wedge\phi+fd\phi \\ d(\phi\wedge\psi) &= d\phi\wedge\psi-\phi\wedge d\psi \end{aligned}$$

More generally, if $\boldsymbol{\xi}$ is a p-form and $\boldsymbol{\eta}$ is a q-form, then

$$\boldsymbol{\xi}\wedge\boldsymbol{\eta} = (-1)^{pq}\boldsymbol{\eta}\wedge\boldsymbol{\xi} \tag{15}$$

$$d(\boldsymbol{\xi}\wedge\boldsymbol{\eta}) = (d\boldsymbol{\xi})\wedge\boldsymbol{\eta}+(-1)^p\boldsymbol{\xi}\wedge(d\boldsymbol{\eta}) \tag{16}$$

DEFINITION. *If $\phi=\sum f_i dx_i$ is a 1-form on $\mathbb{R}^3$, the exterior derivative of ϕ is the 2-form $d\phi=\sum df_i\wedge dx_i$.*

$$d\phi=\left(\frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}\right)dx_1dx_2+\left(\frac{\partial f_3}{\partial x_2}-\frac{\partial f_2}{\partial x_3}\right)dx_2dx_3+\left(\frac{\partial f_1}{\partial x_3}-\frac{\partial f_3}{\partial x_1}\right)dx_3dx_1$$

Let g be a real-valued function on $\mathbb{R}^3$, then we have

$$\begin{aligned} dg &= \sum\frac{\partial g}{\partial x_i}dx_i \\ &= \sum f_i dx_i \\ f_i &= \frac{\partial g_i}{\partial x_i} \end{aligned}$$

then we have

$$\begin{aligned} \left(\frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}\right)dx_1dx_2 &= \left(\frac{\partial}{\partial x_1}\frac{\partial g}{\partial x_2}-\frac{\partial}{\partial x_2}\frac{\partial g}{\partial x_1}\right)dx_1dx_2=0 \\ d(dg) &= \left(\frac{\partial f_2}{\partial x_1}-\frac{\partial f_1}{\partial x_2}\right)dx_1dx_2+\left(\frac{\partial f_3}{\partial x_2}-\frac{\partial f_2}{\partial x_3}\right)dx_2dx_3+\left(\frac{\partial f_1}{\partial x_3}-\frac{\partial f_3}{\partial x_1}\right)dx_3dx_1 \\ d(dg) &= 0 \qquad (17) \end{aligned}$$

The property holds in more generality: in fact, $d(d\alpha)=0$ for any k-form α; more succinctly,$d^2=0$. A $k-$ form ω on U is closed if $d\omega=0$; it's exact if there is a $(k-1)-$ form τ such that $\omega=d\tau$ on U. Since $d(d\tau)=0$, every exact form is closed.

Example. (MAXWELL'S EQUATION) For connection form ω, we have its exterior derivative [1]

$$\Omega=d\omega, d\Omega=d(d\omega)=0$$

When S is 4-dimensional Lorenz manifold, then we have

$$ds^2=-dx_0^2+dx_1^2+dx_2^2+dx_3^2$$

Let $\Omega=\frac{1}{2}\sum F_{ij}dx_i\wedge dx_j$, where $F_{ij}=-F_{ji}$ is a 2-form [5], and the electromagnetic four-potential A is a 1-form including an electric scalar potential and a magnetic vector potential.

$$\begin{aligned} A &\overset{(1)}{\longleftrightarrow} \left(\frac{\phi}{c},\mathbf{A}\right) \\ F &\overset{\text{def}}{=} dA \\ F_{ij} &= \frac{\partial A_i}{\partial x_j}-\frac{\partial A_j}{\partial x_i} \\ F_{ij} &= \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \end{aligned}$$

For ds^2, there is the operator $\star$ such that $d^\star=\star d\star$. If $j=(j_1,j_2,j_3)$, then

$$J = -\rho dx_1dx_2dx_3+dx_0(j_1dx_2dx_3+j_2dx_3dx_1+j_3dx_1dx_2)$$

Since $dF_{ij}\wedge dx_i\wedge dx_j$ is

$$\begin{pmatrix} 0 & dE_1\wedge dx_0\wedge dx_1 & dE_2\wedge dx_0\wedge dx_2 & dE_3\wedge dx_0\wedge dx_3 \\ -dE_1\wedge dx_1\wedge dx_0 & 0 & -dB_3\wedge dx_1\wedge dx_2 & dB_2\wedge dx_1\wedge dx_3 \\ -dE_2\wedge dx_2\wedge dx_0 & dB_3\wedge dx_2\wedge dx_1 & 0 & -dB_1\wedge dx_2\wedge dx_3 \\ -dE_3\wedge dx_3\wedge dx_0 & -dB_2\wedge dx_3\wedge dx_1 & dB_1\wedge dx_3\wedge dx_2 & 0 \end{pmatrix}$$

Moreover

$$\begin{aligned} dE_1\wedge dx_0\wedge dx_1 &= \left(\frac{\partial E_1}{\partial x_0}dx_0+\frac{\partial E_1}{\partial x_1}dx_1+\frac{\partial E_1}{\partial x_2}dx_2+\frac{\partial E_1}{\partial x_3}dx_3\right)\wedge dx_0\wedge dx_1 \\ &= \frac{\partial E_1}{\partial x_2}dx_2\wedge dx_0\wedge dx_1+\frac{\partial E_1}{\partial x_3}dx_3\wedge dx_0\wedge dx_1 \\ &= \frac{\partial E_1}{\partial x_2}dx_0\wedge dx_1\wedge dx_2+\frac{\partial E_1}{\partial x_3}dx_0\wedge dx_1\wedge dx_3 \end{aligned}$$

Similarly we have

$$\begin{aligned} dE_2\wedge dx_0\wedge dx_2 &= -\frac{\partial E_2}{\partial x_1}dx_0\wedge dx_1\wedge dx_2+\frac{\partial E_2}{\partial x_3}dx_0\wedge dx_2\wedge dx_3 \\ dE_3\wedge dx_0\wedge dx_3 &= -\frac{\partial E_3}{\partial x_1}dx_0\wedge dx_1\wedge dx_3-\frac{\partial E_3}{\partial x_2}dx_0\wedge dx_2\wedge dx_3 \\ -dB_3\wedge dx_1\wedge dx_2 &= -\frac{\partial B_3}{\partial x_0}dx_0\wedge dx_1\wedge dx_2-\frac{\partial B_3}{\partial x_3}dx_1\wedge dx_2\wedge dx_3 \\ dB_2\wedge dx_1\wedge dx_3 &= \frac{\partial B_2}{\partial x_0}dx_0\wedge dx_1\wedge dx_3-\frac{\partial B_2}{\partial x_2}dx_1\wedge dx_2\wedge dx_3 \\ -dB_1\wedge dx_2\wedge dx_3 &= -\frac{\partial B_1}{\partial x_0}dx_0\wedge dx_2\wedge dx_3-\frac{\partial B_1}{\partial x_1}dx_1\wedge dx_2\wedge dx_3 \end{aligned}$$

Therefore

$$
\begin{aligned}
d\Omega &= \frac{1}{2} d\left(\sum F_{ij} dx_i \wedge dx_j\right) \\
&= \frac{1}{2} \sum dF_{ij} \wedge dx_i \wedge dx_j \\
&= \left(\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} - \frac{\partial B_3}{\partial x_0}\right) dx_0 \wedge dx_1 \wedge dx_2 + \left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} + \frac{\partial B_2}{\partial x_0}\right) dx_0 \wedge dx_1 \wedge dx_3 + \\
&\quad \left(\frac{\partial E_2}{\partial x_3} - \frac{\partial E_3}{\partial x_2} - \frac{\partial B_1}{\partial x_0}\right) dx_0 \wedge dx_2 \wedge dx_3 + \left(-\frac{\partial B_3}{\partial x_3} - \frac{\partial B_2}{\partial x_2} - \frac{\partial B_1}{\partial x_1}\right) dx_1 \wedge dx_2 \wedge dx_3
\end{aligned}
$$

Because $d\Omega = 0$, thus we have

$$\left(\frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1} - \frac{\partial B_3}{\partial x_0}\right) dx_0 dx_1 dx_2 = 0 \tag{18}$$

$$\left(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} + \frac{\partial B_2}{\partial x_0}\right) dx_0 dx_1 dx_3 = 0 \tag{19}$$

$$\left(\frac{\partial E_2}{\partial x_3} - \frac{\partial E_3}{\partial x_2} - \frac{\partial B_1}{\partial x_0}\right) dx_0 dx_2 dx_3 = 0 \tag{20}$$

and

$$-\frac{\partial B_3}{\partial x_3} - \frac{\partial B_2}{\partial x_2} - \frac{\partial B_1}{\partial x_1} = 0 \Rightarrow \nabla \cdot B = 0 \tag{21}$$

Given $x_0 = t$, $-(18)+(19)-(20)$ is

$$\frac{\partial B}{\partial t} + \begin{vmatrix} dx_0 dx_2 dx_3 & dx_0 dx_1 dx_3 & dx_0 dx_1 dx_2 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ E_1 & E_2 & E_3 \end{vmatrix} = 0$$

$$\frac{\partial B}{\partial t} + \nabla \times E = 0 \tag{22}$$

Using $d^\star \Omega = 4\pi J$, we have

$$\nabla \cdot E = 4\pi\rho \tag{23}$$

$$\nabla \times B - \frac{\partial E}{\partial t} = 4\pi j \tag{24}$$

where $t = x_0$, $E = (E_1, E_2, E_3)$ is the electric field, $B = (B_1, B_2, B_3)$ is the magnetic field, ρ is the charge density, j is the current density. Moreover, $\rho dx_1 \wedge dx_2 \wedge dx_3$ is the charge, $j_1 dx_2 dx_3 + j_2 dx_3 dx_1 + j_3 dx_1 dx_2$ is the electric flux $j \cdot dS$. $dx_0 \wedge (j \cdot dS)$ is the electric current through surface dS. Using $d^2 = 0$, we have $dJ = 0$, which is the law of charge conservation, also known as continuity equation,

$$
\begin{aligned}
dJ &= -d\rho dx_1 dx_2 dx_3 + d(dx_0(j_1 dx_2 dx_3 + j_2 dx_3 dx_1 + j_3 dx_1 dx_2)) \\
&= -d\rho dx_1 dx_2 dx_3 + d(j_1 dx_0 dx_2 dx_3 + j_2 dx_0 dx_3 dx_1 + j_3 dx_0 dx_1 dx_2) \\
&= -\frac{\partial \rho}{\partial x_0} dx_0 dx_1 dx_2 dx_3 + dj_1 dx_0 dx_2 dx_3 - dj_2 dx_0 dx_1 dx_3 + dj_3 dx_0 dx_1 dx_2 \\
&= -\frac{\partial \rho}{\partial t} dx_0 dx_1 dx_2 dx_3 + \frac{\partial j_1}{\partial x_1} dx_1 dx_0 dx_2 dx_3 - \frac{\partial j_2}{\partial x_2} dx_2 dx_0 dx_1 dx_3 + \frac{\partial j_3}{\partial x_3} dx_3 dx_0 dx_1 dx_2 \\
&= -\left(\frac{\partial \rho}{\partial t} + \frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3}\right) dx_0 dx_1 dx_2 dx_3 \\
&= -\left(\frac{\partial \rho}{\partial t} + \nabla \cdot j\right) dx_0 dx_1 dx_2 dx_3
\end{aligned}
$$

Thus we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

1.5. Tangent Map

Let $F = (f_1, f_2, ..., f_m)$ be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. If $\mathbf{v}_p$ is a tangent vector to $\mathbb{R}^n$ at p, then

$$F_*(\mathbf{v}) = (\mathbf{v}[f_1], ..., \mathbf{v}[f_m]) \text{ at } F(p) \tag{25}$$

If $\beta = F(\alpha(t))$ is the image of a curve α in $\mathbb{R}^n$, then $\beta' = F_*(\alpha')$.

1.6. Frame Fields

THEOREM. *If $\beta: I \to \mathbb{R}^3$ is a unit-speed curve with curvature $\kappa > 0$ and torsion τ, then*

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \tag{26}$$

DEFINITION. *If E_i is a frame on $\mathbb{R}^3, v \in T_p(\mathbb{R}^3)$, then $v = \sum v_i E_i$, define 1-form $\theta_i: T_p(\mathbb{R}^3) \to \mathbb{R}$, $\theta_i(v) = v_i$. $\theta_1, \theta_2, \theta_3$ are the dual 1-forms of E_1, E_2, E_3.*

DEFINITION. *Attitude matrix A connects two frame fields as*

$$E = AU \tag{27}$$

where $A = [a_{ij}] \in \mathrm{SO}_n$.

DEFINITION. *Connection form ω is defined in*

$$\nabla_v E = \omega E \tag{28}$$

Because $\nabla_v E_i = \sum_{j=1}^3 \omega_{ij} E_j$, we have

$$\begin{aligned} \omega_{ij} &= (\nabla_v E_i) \cdot E_j \\ &= \left(\sum_{k=1}^3 v[a_{ik}] U_k \right) \cdot \left(\sum_{l=1}^3 a_{jl} U_l \right) \\ &= \sum_{k=1}^3 v[a_{ik}] a_{jk} \\ &= \sum_{k=1}^3 d a_{ik}(v) a_{jk} \end{aligned}$$

In matrix form, it's written as

$$\omega = (dA) A^T \tag{29}$$

ω is a skew symmetric matrix. Because

$$\begin{aligned} A A^T &= I \\ d(A A^T) &= 0 \\ (dA) A^T + A (dA^T) &= 0 \\ (dA) A^T + ((dA) A^T)^T &= 0 \\ (dA) A^T &= -((dA) A^T)^T \\ \omega &= -\omega^T \end{aligned} \tag{30}$$

Example. Find the connection form of the cylindrical frame field.

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

So we have

$$A^T = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad dA = \begin{pmatrix} -\sin(\theta) d\theta & \cos(\theta) d\theta & 0 \\ -\cos(\theta) d\theta & -\sin(\theta) d\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The connection form is

$$\omega = (dA)A^T = \begin{pmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{31}$$

DEFINITION. *Denoting the vector space of all linear maps $f: V \to W$, where V, W are real vector spaces, the dual space of V is $V^\vee = \mathrm{Hom}(V, \mathbb{R})$.*

DEFINITION. *If E_i is a frame field, then the dual 1-forms θ_i of the frame field are the 1-forms s.t.*

$$\theta_i(v) = v \cdot E_i = v_i \tag{32}$$

for each tangent vector v to $\mathbb{R}^3$ at p. It satisfies

$$\theta_i(E_j) = E_i \cdot E_j = \delta_{ij} \tag{33}$$

LEMMA. *Any 1-form ϕ on $\mathbb{R}^3$ in a frame field E_i has a unique expression*

$$\phi = \sum \phi(E_i)\theta_i \tag{34}$$

Proof. Apply to any tagent vector v,

$$\begin{aligned} \left(\sum \phi(E_i)\theta_i\right)(v) &= \sum \phi(E_i)\theta_i(v) \\ &= \sum v_i \phi(E_i) \\ &= \phi\left(\sum v_i E_i\right) \\ &= \phi(v) \end{aligned}$$

Generally $\sum \phi(E_i)\theta_i = \phi$. □

Example. In particular, if we choose $E_i = U_i, \theta_i = dx_i$, then by (34),

$$\phi = \sum \phi(U_i)dx_i \tag{35}$$

for $E_i = \sum a_{ij}U_j$, the dual formulation is just $\theta_i = \sum a_{ij}dx_j$. This is because,

$$\begin{aligned} \theta_i(U_j) &= U_j \cdot E_i \\ &= U_j \cdot \left(\sum a_{ik}U_k\right) \\ &= \sum a_{ik}\delta_{kj} \\ &= a_{ij} \end{aligned}$$

From (35) we have

$$\theta_i = \sum \theta_i(U_j)dx_j = \sum a_{ij}dx_j$$

or in matrix form

$$\theta = A d\xi \tag{36}$$

where $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, d\xi = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$.

PROPOSITION. *The functions $\theta_1, \theta_2, \ldots, \theta_n$ form a basis for $V^\vee$. The basis $\theta_1, \theta_2, \ldots, \theta_n$ for $V^\vee$ is said to be dual to the basis $E_1, E_2, \ldots, E_n$ for V.*

THEOREM. *Cartan structural equations*

$$d\theta_i = \sum_j \omega_{ij} \wedge \theta_j \tag{37}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} \tag{38}$$

Proof. By (36), then

$$\begin{aligned} d\theta &= d(A d\xi) \\ &= (dA)\wedge(d\xi) + A d(d\xi) \\ &= ((dA)A^T)(A d\xi) \\ &= \omega\theta \end{aligned}$$

Use $\omega = (dA)A^T$, then

$$\begin{aligned} d\omega &= d((dA)A^T) \\ &= d(A^T(dA)) \\ &= (d(A^T))(dA) \\ &= -(dA)(d(A^T)) \\ &= ((dA)A^T)(-A d(A^T)) \\ &= \omega(-\omega^T) \\ &= \omega\omega \end{aligned}$$

□

2. Calculus on a Surface

2.1. Differential Forms on a Surface

A 0-form f on a surface M is simply a differentiable real-valued function on M, and 1-form ϕ on M is a real-valued function on tangent vectors to M that is linear at each point. A 2-form η on a surface M is a real-valued function on all ordered pairs of tangent vectors v, w to M such that

1. $\eta(v, w)$ is linear in v and in w;
2. $\eta(v, w) = -\eta(w, v)$.

According to the definition, $\eta(v, v) = 0$. The 2-form satisfies

$$\eta(a\boldsymbol{v} + b\boldsymbol{w}, c\boldsymbol{v} + d\boldsymbol{w}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \eta(\boldsymbol{v}, \boldsymbol{w}) = (ad - bc)\eta(\boldsymbol{v}, \boldsymbol{w}) \tag{39}$$

the wedge product of two 1-forms $\boldsymbol{\phi} \wedge \boldsymbol{\psi}$ is the 2-form on M such that

$$(\boldsymbol{\phi} \wedge \boldsymbol{\psi})(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{\phi}(\boldsymbol{v})\boldsymbol{\psi}(\boldsymbol{w}) - \boldsymbol{\phi}(\boldsymbol{\omega})\boldsymbol{\psi}(\boldsymbol{v})$$

for all pairs $\boldsymbol{v}, \boldsymbol{w}$ of tangent vectors to M. Let r^1, r^2 be standard coordinates on $\mathbb{R}^2$, and let D be an open set in $\mathbb{R}^2$. If $\mathbf{x}: D \to M$ is a proper patch for a surface M and $U = x(D)$, let $\mathbf{x}^{-1} = (x^1, x^2)$: $U \to \mathbb{R}^2$, where x^1, x^2 are the two function components of $\mathbf{x}^{-1}$ and we can write $x^i = r^i \circ \mathbf{x}^{-1}$. We call U a coordinate open set and x^1, x^2 coordinate functions on U. Define the tangent vectors $\partial / \partial x^i$ at $p = \mathbf{x}(u_0, v_0)$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p = \mathbf{x}_*\left(\left.\frac{\partial}{\partial r^i}\right|_{(u_0, v_0)}\right)$$

where $\partial / \partial x^1 = \mathbf{x}_*(U_1) = \mathbf{x}_u$, $\quad \partial / \partial x^2 = \mathbf{x}_*(U_2) = \mathbf{x}_v$. The partial derivative of f with respect to the coordinate x^i can be calculated via bringing it back to $\mathbb{R}^2$:

$$\frac{\partial}{\partial x^i} f = \left(\mathbf{x}_* \frac{\partial}{\partial r^i}\right)(f) = \frac{\partial}{\partial r^i}(f \circ \mathbf{x}) \tag{40}$$

The 1-forms dx^1, dx^2 are dual to the tangent vectors $\partial / \partial x^1, \partial / \partial x^2$ at every point of U. Therefore, every 1-form ϕ on the surface M is $\sum f_i dx^i$ on U and every 2-form on the surface is $f dx^1 \wedge dx^2$ on U for some functions f_i and f on U. The exterior derivative of a 1-form $\phi = \sum f_i dx^i$ is

$$d\phi = \sum df_i \wedge dx^i \tag{41}$$

This definition depends on the choice of a coordinate patch $\mathbf{x}$, but it can be shown that it's in fact independent of coordinate patches.

2.2. Pullback and Pushforward

F^*g is the function on M such that

$$F^*g = g \circ F \tag{42}$$

If ϕ is a 1-form on N, $F^*\phi$ is the 1-form on M such that

$$(F^*\phi)(\boldsymbol{v}) = \phi(F_*\boldsymbol{v}) \tag{43}$$

If η is a 2-form on N, let $F^*\eta$ be the 2-form on M such that

$$(F^*\eta)(\boldsymbol{v}, \boldsymbol{w}) = \eta(F_*\boldsymbol{v}, F_*\boldsymbol{w}) \tag{44}$$

Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be forms on N, the pullback operation satisfies

$$\begin{aligned} F^*(\boldsymbol{\xi} + \boldsymbol{\eta}) &= F^*\boldsymbol{\xi} + F^*\boldsymbol{\eta} \\ F^*(\boldsymbol{\xi} \wedge \boldsymbol{\eta}) &= F^*\boldsymbol{\xi} \wedge F^*\boldsymbol{\eta} \\ F^*(d\boldsymbol{\xi}) &= d(F^*\boldsymbol{\xi}) \end{aligned}$$

Let $F: M \to N$ be a mapping of surfaces. If g is a 0-form on N, F_*g is the function on M such that

$$F_*g = g \circ F^{-1} \tag{45}$$

Proof. Choose a point p on M, then

$$\begin{aligned} F_*(g(p)) &= g(p) \\ &= g \circ F^{-1}(F(p)) \\ F_*g &= g \circ F^{-1} \end{aligned}$$

Further we have $F_*(gv) = g \circ F^{-1} F_*(v)$. However, if g is a function on the curve $g = g(t)$, then $F_*g = g$. □

2.3. Integration of Forms

Let ϕ be a 1-form on M, and let $\alpha: [a, b] \to M$ be a curve segment on a surface M. Then

$$\int_\alpha \phi = \int_{[a,b]} \alpha^*\phi = \int_a^b \phi(\alpha'(t)) dt \tag{46}$$

Particularly we have

$$\int_\alpha df = f(\alpha(b)) - f(\alpha(a)) \tag{47}$$

Example. If $f = uv^2$, $\phi = df = v^2 du + 2uv dv$, and α is the curve segment given by $\alpha(t) = (t, t^2)$, then we have

$$\begin{aligned} \alpha'(t) &= (1, 2t) \\ \int_\alpha \phi &= \int_a^b \phi(\alpha'(t)) dt \\ &= \int_{-1}^1 (t^2)^2 du\,(\alpha'(t)) + 2t * t^2 dv(\alpha'(t)) dt \\ &= \int_{-1}^1 (t^4 * 1 + 2t * t^2 * 2t\,) dt \\ &= \int_{-1}^1 (5t^4) dt \\ &= t^5|_{-1}^1 \\ &= 2 \\ &= f(\alpha(1)) - f(\alpha(-1)) \end{aligned}$$

Let η be a 2-form on M, and let $\mathbf{x}\colon[a,b]\times[c,d]\to M$ be a 2-segment (differentiable but need not be 1-1 or regular) on a surface M. Then

$$\int_{\mathbf{x}}\eta=\int\int_R \mathbf{x}^*\eta=\int_a^b\int_c^d \eta(\mathbf{x}_u,\mathbf{x}_v)dt \tag{48}$$

THEOREM. (STOKES' THEOREM) *If ϕ is a 1-form on M, and $\mathbf{x}\colon[a,b]\times[c,d]\to M$ is a 2-segment,*

$$\int_{\mathbf{x}} d\phi=\int_{\partial\mathbf{x}}\phi \tag{49}$$

where $\int_{\partial\mathbf{x}}\phi=\int_\alpha\phi+\int_\beta\phi-\int_\gamma\phi-\int_\delta\phi$.

By convention, if $k\neq k'$, the integral of a $k-$dimensional form on a $k'-$dimensional surface is understood to be zero [3].

It should be noted that

$$\iint_A f dx\wedge dy=\iint_A f dx dy=\iint_A f dy dx=-\iint_A f dy\wedge dx \tag{50}$$

2.4. Topological Properties of Surfaces

DEFINITION. *M is path-connected if any two points on M can be joined by a path.*

DEFINITION. *M is compact if every open cover of M has a finite subcover.*

Example. An open cover $\left\{\left(\frac{1}{n},1\right)\right\}_{n=2}^{\infty}$ does not have a finite subcover. So $(0,1)$ is not compact.

THEOREM. *A subset of $\mathbb{R}^n$ is compact iff it's closed and bounded.*

Example. A sphere is closed and bounded, so it's compact.

THEOREM. *A continuous function on a compact space attains a maximum and a minimum.*

Example. A Cylinder $S^1\times(-1,1)$ is not compact, so there is no maximum.

DEFINITION. *A surface is orientable if there is a 2-form η on M that's never 0 at any point.*

PROPOSITION. *A surface M is orientable iff it has a continuous unit normal vector field.*

Proof. Let $U(p)$ be a continuous unit normal vector field for $p\in M$. Define $\phi_p(v,w)=U_p\cdot(v\times w)=\det[U_p,\ v,\ w]$ is bilinear in v, w and alternating. If v, w are independent, then U_p, v, w are independent and $\phi_p(v,w)\neq 0$. □

3. SHAPE OPERATORS

3.1. Shape Operators of a Surface

DEFINITION. *If $\mathbf{p}$ is a point of M, then for each tangent vector $\mathbf{v}$ to M at $\mathbf{p}$, let*

$$S_p(\mathbf{v})=-\nabla_{\mathbf{v}}U$$

where U is a unit normal vector field on a neighborhood of $\mathbf{p}$ in M. S_p is called the shape operator of M at $\mathbf{p}$ derived from U.

LEMMA. *For each point $\mathbf{p}$ of $M\subset\mathbb{R}^3$, the shape operator is a linear operator*

$$S_p\colon T_p(M)\to T_p(M)$$

on the tangent plane of M at $\mathbf{p}$.

Proof. Use $U \cdot U = 1$, and differentiate both sides, we have

$$0 = \mathbf{v}[U \cdot U] = 2(\nabla_{\mathbf{v}} U) \cdot U = -2S_p(\mathbf{v}) \cdot U$$

Thus $S_p(\mathbf{v}) \in T_p(M)$. □

Example. Given a sphere $x^2 + y^2 + z^2 = r^2$, we have

$$U = \frac{1}{r}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \nabla_{\mathbf{v}} U = \frac{1}{r}\begin{pmatrix} \mathbf{v}[x] \\ \mathbf{v}[y] \\ \mathbf{v}[z] \end{pmatrix} = \frac{1}{r}\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix} = \frac{1}{r}\mathbf{v}$$

Let $\mathbf{v} = \sum \mathbf{v}_i U_i = \sum \mathbf{v}_i \frac{\partial}{\partial x_i}$, then

$$S_p(\mathbf{v}) = -\nabla_{\mathbf{v}} U = -\frac{1}{r}\mathbf{v} \tag{51}$$

This is represented by $\begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}$.

Example. For a plane $S_p(\mathbf{v}) = -\nabla_{\mathbf{v}} U = 0$, because U is constant, it is represented by $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Example. Given a cylinder $x^2 + y^2 = 1$, we have $U = \frac{1}{r}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$.

$$S_p(\mathbf{v}) = -\nabla_{\mathbf{v}} U = -\frac{1}{r}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

A basis for $T_p(M)$ is $\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \theta}\right\}$, where $\frac{\partial}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\frac{\partial}{\partial \theta} = \frac{1}{r}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$, then

$$S_p\left(\frac{\partial}{\partial z}\right) = 0, \quad S_p\left(\frac{\partial}{\partial \theta}\right) = -\frac{1}{r}\begin{pmatrix} -\frac{y}{r} \\ \frac{x}{r} \\ 0 \end{pmatrix} = -\frac{1}{r}\left(\frac{\partial}{\partial \theta}\right)$$

This is the matrix $\begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}$.

THEOREM. *Relative to any orthonormal basis of $T_p(M)$, S_p is represented by a 2×2 symmetric matrix.*

Proof. If e_1, e_2 is a basis for $T_p(M)$, then let

$$\begin{pmatrix} S_p(e_1) \\ S_p(e_2) \end{pmatrix} = \begin{pmatrix} a & d \\ b & c \end{pmatrix}^T \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

Consider that the basis x_u, x_v is tangent to M, so $U \cdot x_u = 0$. Differentiate with respect to v,

$$\begin{aligned} \frac{\partial U}{\partial v} \cdot x_u + U \cdot \frac{\partial x_u}{\partial v} &= 0 \\ \left(\frac{d}{dv}\bigg|_{v=v_0} U(x(u_0, v))\right) \cdot x_u + U \cdot x_{uv} &= 0 \\ \nabla_{x_v} U \cdot x_u + U \cdot x_{uv} &= 0 \\ -S_p(x_v) \cdot x_u + U \cdot x_{uv} &= 0 \end{aligned}$$

Further we have

$$S_p(x_v) \cdot x_u = U \cdot x_{uv} = U \cdot x_{vu} = S_p(x_u) \cdot x_v \tag{52}$$

1. Dot product is the first foundamental bilinear form;

2. $S_p(x_v) \cdot x_u$ is the second foundamental bilinear form.

Now suppose

$$e_1 = f x_u + g x_v \quad e_2 = h x_u + j x_v$$

is any orthonormal basis. Then

$$\begin{aligned} b &= S(e_1) \cdot e_2 \\ &= S(f x_u + g x_v) \cdot (h x_u + j x_v) \\ &= f h S(x_u) \cdot x_u + g j S(x_v) \cdot x_v + (f j + g h)\, S(x_u) \cdot x_v \\ &= S(e_2) \cdot e_1 \\ &= d \end{aligned}$$

Thus, S_p is represented by a 2×2 symmetric matrix. □

THEOREM. *The eigenvectors of a symmetric matrix A corresponding to two distinct eigenvalues λ_1, λ_2 are orthogonal.*

Proof. We know $A v_1 = \lambda_1 v_1, A v_2 = \lambda_2 v_2$, and $A = A^T$, then

$$\begin{aligned} (\lambda_1 v_1) \cdot v_2 &= (A v_1) \cdot v_2 \\ &= (A v_1)^T v_2 \\ &= v_1^T A^T v_2 \\ &= v_1^T (A v_2) \\ &= \lambda_2 v_1^T v_2 \\ &= \lambda_2 v_1 \cdot v_2 \\ (\lambda_1 - \lambda_2) v_1 \cdot v_2 &= 0 \end{aligned}$$

Since $\lambda_1 - \lambda_2 \neq 0$, then $v_1 \cdot v_2 = 0$, which means two eigenvectors are orthogonal. □

3.2. Normal Curvature

DEFINITION. *Let M be a surface with unit normal vector field U, and let $\mathbf{p} \in M$, $u \in T_p(M)$,* and $\|u\| = 1$*, then the normal curvature is*

$$k(u) = \alpha''(s) \cdot U \tag{53}$$

where $\alpha(s)$ is a curve parameterized by arclength with $\alpha(0) = \mathbf{p}, \alpha'(0) = u$.

Because $\alpha'(s)$ is tangent to M, $\alpha'(s) \cdot U_{\alpha(s)} = 0$. Differentiate with respect to s:

$$\alpha''(s) \cdot U_{\alpha(s)} + \alpha'(s) \cdot \frac{d}{ds} U_{\alpha(s)} = 0$$

Evaluate at $s = 0$, we have

$$\begin{aligned} \alpha''(0) \cdot U_{\alpha(0)} &= -\alpha'(0) \cdot \left.\frac{d}{ds}\right|_{s=0} U_{\alpha(s)} \\ \alpha''(0) \cdot U_{\alpha(0)} &= -\alpha'(0) \cdot \nabla_{\alpha'(0)} U \\ \alpha''(0) \cdot U_{\alpha(0)} &= u \cdot S_p(U) \end{aligned} \tag{54}$$

which is the normal curvature at $\mathbf{p}$ in the direction u. Suppose $u = \begin{pmatrix} x \\ y \end{pmatrix}$, and $x^2 + y^2 = 1$, and $S_p = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ relates to the same basis and it's symmetric. The normal curvature is

$$\begin{aligned} k(u) &= u \cdot S_p(u) \\ &= \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= a x^2 + 2 b x y + c y^2 \end{aligned}$$

By lagrange, the max and the min occur when

$$\begin{aligned} \nabla k &= \lambda_1 \nabla g \\ \begin{pmatrix} 2ax+2by \\ 2bx+2cy \end{pmatrix} &= \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \lambda \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

The max and the min occur at the 2 eigenvectors

$$k(x,y) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda(x^2+y^2) = \lambda$$

The max and the min normal curvature are the eigenvalues of S_p.

DEFINITION. *Gauss curvature* $K(p) = \det(S_p) = \lambda_1\lambda_2$.

DEFINITION. *Mean curvature* $H(p) = \frac{1}{2}(\lambda_1+\lambda_2) = \frac{1}{2}\mathrm{tr}(S_p)$.

DEFINITION. *Principal curvatures are* λ_1, λ_2*; principal directions are the orthogonal eigenvectors. They are orthogonal because* S_p *is symmetric.*

Example. Let (x_p, y_p, z_p) be any point $\mathbf{p}$ of a surface function $z = f(x,y)$, then the tangent plane is

$$z = f(x_p, y_p) + f_x(x_p, y_p)(x - x_p) + f_y(x_p, y_p)(y - y_p) \tag{55}$$

The unit normal vector is

$$U = \pm \frac{1}{\sqrt{1+f_x^2+f_y^2}} \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}\Bigg|_{(x_p, y_p)} \tag{56}$$

The unit normal vector to a plane specified by

$$F(x,y,z) = 0$$

is given by

$$U = \pm \frac{\nabla F}{\sqrt{F_x^2+F_y^2+F_z^2}}$$

Specifically for $F(x,y,z) = ax+by+cz+d$, we have $U = \pm\nabla f = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

4. GEOMETRY OF SURFACES IN $\mathbb{R}^3$

4.1. Structural Equations for Surfaces

DEFINITION. *An adapted frame field on* M *is a triple of orthonormal Euclidean vector field* (E_1, E_2, E_3) *on surface* M *such that* E_3 *is normal to* M*, and so* E_1, E_2 *are tangent to* M*.*

When we restrict the structural equations to surface M, for all $v \in T_p(M)$,

$$\theta_3(v) = v \cdot E_3 = 0$$

The first structural equation becomes

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \\ 0 \end{pmatrix} \tag{57}$$

Further simplification generates

$$d\theta_1 = \omega_{12}\theta_2 \tag{58}$$

$$d\theta_2 = \omega_{21}\theta_1 \tag{59}$$

$$\omega_{31}\wedge\theta_1 + \omega_{32}\wedge\theta_2 = 0 \tag{60}$$

The second structural equation becomes

$$
\begin{aligned}
d\omega_{12} &= \omega_{13}\wedge\omega_{32} && (61)\\
d\omega_{13} &= \omega_{12}\wedge\omega_{23} && (62)\\
d\omega_{23} &= \omega_{21}\wedge\omega_{13} && (63)
\end{aligned}
$$

The shape operator here is

$$
\begin{aligned}
\nabla_v E_3 &= \omega_{31}(v)E_1+\omega_{32}(v)E_2+\omega_{33}(v)E_3\\
S(v) &= -\nabla_v E_3\\
&= \omega_{13}(v)E_1+\omega_{23}(v)E_2
\end{aligned}
$$

Relative to E_1, E_2, then

$$
\begin{aligned}
S(E_1) &= \omega_{13}(E_1)E_1+\omega_{23}(E_1)E_2\\
S(E_2) &= \omega_{13}(E_2)E_1+\omega_{23}(E_2)E_2
\end{aligned}
$$

Matrix representation of S is

$$
S=\begin{pmatrix} \omega_{13}(E_1) & \omega_{13}(E_2)\\ \omega_{23}(E_1) & \omega_{23}(E_2)\end{pmatrix}
$$

Gaussian curvature is

$$
\begin{aligned}
K &= |S|\\
&= \omega_{13}(E_1)\omega_{23}(E_2)-\omega_{13}(E_2)\omega_{23}(E_1)\\
&= (\omega_{13}\wedge\omega_{23})(E_1,E_2)\\
&= -d\omega_{12}(E_1,E_2)
\end{aligned}
$$

Since $\omega_{13}\wedge\omega_{23}$ is a 2-form, and $\theta_1\wedge\theta_2$ is a 2-form on M as well, then suppose $\omega_{13}\wedge\omega_{23}=f\theta_1\wedge\theta_2$, and apply both sides to E_1, E_2,

$$
\begin{aligned}
\omega_{13}\wedge\omega_{23}(E_1,E_2) &= f\theta_1\wedge\theta_2(E_1,E_2)\\
K &= f
\end{aligned}
$$

Therefore,

$$
\omega_{13}\wedge\omega_{23}=K\theta_1\wedge\theta_2=-d\omega_{12} \qquad (64)
$$

For the mean curvature, use

$$
\begin{aligned}
2H &= \omega_{13}(E_1)+\omega_{23}(E_2)\\
&= \omega_{13}(E_1)\theta_2(E_2)-\omega_{13}(E_2)\theta_2(E_1)+\theta_1(E_1)\omega_{23}(E_2)-\theta_1(E_2)\omega_{23}(E_1)\\
&= (\omega_{13}\wedge\theta_2+\theta_1\wedge\omega_{23})(E_1,E_2)
\end{aligned}
$$

Suppose $\omega_{13}\wedge\theta_2+\theta_1\wedge\omega_{23}=f\theta_1\wedge\theta_2$, and apply both sides to E_1, E_2, then $f=2H$. And

$$
\omega_{13}\wedge\theta_2+\theta_1\wedge\omega_{23} = 2H\theta_1\wedge\theta_2 \qquad (65)
$$

Example. Let M be a sphere of radius ρ, then $x=\rho\cos\phi\cos\theta, y=\rho\cos\phi\sin\theta, z=\rho\sin\phi$, therefore

$$
\frac{\partial}{\partial\phi}=\begin{pmatrix}-\rho\sin\phi\cos\theta\\ -\rho\sin\phi\sin\theta\\ \rho\cos\phi\end{pmatrix} \qquad \frac{\partial}{\partial\theta}=\begin{pmatrix}-\rho\cos\phi\sin\theta\\ \rho\cos\phi\cos\theta\\ 0\end{pmatrix}
$$

Choosing the adapted frame as $E_1=\frac{1}{\rho}\frac{\partial}{\partial\phi}, E_2=\frac{1}{\rho\cos\phi}\frac{\partial}{\partial\theta}$, the dual forms are $\theta_1=\rho d\phi, \theta_2=\rho\cos\phi d\theta$. Let $\omega_{12}=a d\phi+b d\theta$. Here ρ is constant, and θ, ϕ are functions. The first structural equation gives

$$
\begin{aligned}
d\theta_1 &= \omega_{12}\theta_2\\
0 &= (a d\phi+b d\theta)\rho\cos\phi d\theta\\
0 &= a\rho\cos\phi d\phi\wedge d\theta\\
a &= 0
\end{aligned}
$$

use another first structural equation

$$\begin{aligned} d\theta_2 &= -\omega_{12}\theta_1 \\ -\rho\sin\phi d\phi \wedge d\theta &= -(a\,d\phi + b\,d\theta)\rho d\phi \\ -\rho\sin\phi d\phi \wedge d\theta &= b\rho d\phi \wedge d\theta \\ b &= -\sin\phi \end{aligned}$$

Thus $\omega_{12} = -\sin\phi d\theta$, to calculate the Gaussian curvature via (64),

$$K = -\frac{d\omega_{12}}{\theta_1 \wedge \theta_2} = -\frac{-\cos\phi d\phi \wedge d\theta}{(\rho d\phi) \wedge (\rho\cos\phi d\theta)} = \frac{1}{\rho^2} \tag{66}$$

4.2. Isometries

The intrinsic distance between p and q on surface M is

$$\rho(p,q) := \inf L(\alpha) = \inf \int_a^b \|\alpha'(t)\| dt \tag{67}$$

An isometry $f: M \to \bar{M}$ is a bijective differentiable map such that

$$(f_* v) \cdot (f_* w) = v \cdot w$$

for all $v, w \in T_p(M)$. TFAE:

1. $(f_* v) \cdot (f_* w) = v \cdot w$;
2. $\|f(v)\| = \|v\|$;
3. f preserves orthonormal basis

A map $f: M \to \bar{M}$ is a local isometry if it preserves the dot product.

DEFINITION. *A property of a surface that is invariant under isometries is intrinsic.*

THEOREM. *Gaussian curvature K is intrinsic.*

Proof. 1st structional equation on $\bar{M}$:

$$d\bar{\theta}_1 = \overline{\omega_{12}} \wedge \bar{\theta}_2, \quad d\bar{\theta}_2 = \overline{\omega_{21}} \wedge \bar{\theta}_1$$

Take F^* for both sides,

$$\begin{aligned} F^* d\bar{\theta}_1 &= F^*(\overline{\omega_{12}} \wedge \bar{\theta}_2) \\ dF^*\bar{\theta}_1 &= F^*\overline{\omega_{12}} \wedge F^*\bar{\theta}_2 \\ d\theta_1 &= F^*\overline{\omega_{12}} \wedge \theta_2 \end{aligned}$$

Similarly we have

$$d\theta_2 = F^*\overline{\omega_{21}} \wedge \theta_1$$

By Cartan's lemma, $F^*\overline{\omega_{12}} = \omega_{12}$. Use Gauss's equation on $\bar{M}$,

$$\begin{aligned} d\overline{\omega_{12}} &= -\bar{K}\bar{\theta}_1 \wedge \bar{\theta}_2 \\ F^* d\overline{\omega_{12}} &= -F^*\bar{K} F^*\bar{\theta}_1 \wedge F^*\bar{\theta}_2 \\ dF^*\overline{\omega_{12}} &= -F^*\bar{K}\theta_1 \wedge \theta_2 \\ d\omega_{12} &= -\bar{K} \circ F\theta_1 \wedge \theta_2 \end{aligned}$$

Therefore

$$K = \bar{K} \circ F$$

At any point $p \in M$,

$$\begin{aligned} K(p) &= \bar{K} \circ F(p) \\ K(p) &= \bar{K}(F(p)) \end{aligned}$$

The Gaussian curvature K at p is the same as $\bar{K}$ at $F(p)$. □

DEFINITION. *A mapping of surfaces $F\colon M \to N$ is conformal provided there exists a real-valued function $\lambda > 0$ on M such that*

$$\|F_*(v_p)\| = \lambda(p)\|v_p\| \tag{68}$$

for all tangent vectors to M. The function λ is called the scale factor of F. A conformal mapping preserves angles. When $\lambda = 1$, F is a local isometry.

5. Riemannian Geometry

5.1. Geometric Surfaces

DEFINITION. *An inner product on a vector space V is a function $<,>: V \times V \to \mathbb{R}$ that has these 3 properties: bilinearity; symmetry; postive definitness.*

Example. Conformal change $<v, w> = \frac{v \cdot w}{h^2}$.

DEFINITION. *A geometric surface is a surface M with an inner product on $T_p(M)$ for each $p \in M$ s.t. if X, Y are differentiable vector fields on M, then $<X, Y>$ is a differentiable function.*

DEFINITION. *A frame field on a geometric surface is a pair of orthonormal vector field E_1, E_2. Their dual 1-forms θ_1, θ_2 are 1-forms on M s.t. $\theta_i(E_j) = \delta_{ij}$, or $\theta_i(v) = <v, E_i>$.*

Let $\overline{E_1}, \overline{E_2}$ be another frame field on M, and

$$\begin{aligned}
\begin{pmatrix} \overline{E_1} & \overline{E_2} \end{pmatrix} &= \begin{pmatrix} E_1 & E_2 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\
\bar{E} &= EA \\
\bar{\theta}\bar{E} &= \begin{pmatrix} \bar{\theta_1} \\ \bar{\theta_2} \end{pmatrix}\begin{pmatrix} \overline{E_1} & \overline{E_2} \end{pmatrix} \\
&= I \\
\begin{pmatrix} \bar{\theta_1} \\ \bar{\theta_2} \end{pmatrix} &= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \\
\bar{\theta} &= B\theta \\
\bar{\theta}\bar{E} &= B\theta EA \\
&= BA \\
&= I \\
B &= A^{-1} \\
&= A^T
\end{aligned} \tag{69}$$

The relation between $\bar{\omega}$ and ω:

$$\begin{aligned}
d\bar{\theta} &= d(B\theta) \\
&= (dB)\theta + Bd\theta \\
&= (dB)B^{-1}\bar{\theta} + B\omega\theta \\
&= (dBB^{-1} + B\omega B^{-1})\bar{\theta} \\
\bar{\omega} &= dBB^{-1} + B\omega B^{-1} \\
&= (dA^T)A + A^T\omega A \\
&= \omega + A^T\omega A
\end{aligned} \tag{70}$$

The area form is

$$\begin{aligned}
\bar{\theta}_1 \wedge \bar{\theta}_2 &= (b_{11}\theta_1 + b_{12}\theta_2) \wedge (b_{21}\theta_1 + b_{22}\theta_2) \\
&= (b_{11}b_{22} - b_{12}b_{21})\theta_1 \wedge \theta_2 \\
&= (\det B)\theta_1 \wedge \theta_2 \\
&= (\det A^T)\theta_1 \wedge \theta_2 \\
&= (\det A)\theta_1 \wedge \theta_2 \\
\bar{\theta}_1 \wedge \bar{\theta}_2 &= \pm\theta_1 \wedge \theta_2
\end{aligned}$$

1. If $\overline{E_1}, \overline{E_2}$ has the same orientation as E_1, E_2,

$$\begin{aligned}
A &= \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \\
\omega &= (dA^T)A \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d\varphi \\
A^T\omega A &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \omega_{12} \\
\bar{\omega} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\omega_{12}} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\omega_{12} + d\varphi) \\
\overline{\omega_{12}} &= \omega_{12} + d\varphi \qquad (71) \\
d\overline{\omega_{12}} &= d\omega_{12} \\
\bar{\theta}_1 \wedge \bar{\theta}_2 &= \theta_1 \wedge \theta_2 \\
\bar{K} &= K
\end{aligned}$$

2. Similarly if $\overline{E_1}, \overline{E_2}$ has the opposite orientation as E_1, E_2,

$$\begin{aligned}
A &= \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & -\cos\varphi \end{pmatrix} \\
\overline{\omega_{12}} &= -(\omega_{12} + d\varphi) \qquad (72) \\
d\overline{\omega_{12}} &= -d\omega_{12} \\
\bar{\theta}_1 \wedge \bar{\theta}_2 &= -\theta_1 \wedge \theta_2 \\
\bar{K} &= K
\end{aligned}$$

K is independent of the choice of frame fields, it's defined as the Gaussian curvature of the geometric surface.

Example. Poincare half-plane

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\} \quad \text{with} \quad < v, w >_{(x,y)} = \frac{v \cdot w}{y^2}$$

where $v, w \in T_{(x,y)}(\mathbb{H}^2) \simeq \mathbb{R}^2$.

$$\begin{aligned}
<\frac{\partial}{\partial x}, \frac{\partial}{\partial x}>_{(x,y)} &= \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{y^2} \\
&= \frac{1}{y^2} \\
<\frac{\partial}{\partial y}, \frac{\partial}{\partial y}>_{(x,y)} &= \frac{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{y^2} \\
&= \frac{1}{y^2}
\end{aligned}$$

$E_1 = y\frac{\partial}{\partial x}, E_2 = y\frac{\partial}{\partial y}$ is a frame field. The dual 1-forms are $\theta_1 = \frac{1}{y}dx, \theta_2 = \frac{1}{y}dy$. The first structional equations are

$$\begin{aligned} d\theta_1 &= -\frac{1}{y^2}dy \wedge dx \\ &= \frac{1}{y}dx \wedge \frac{1}{y}dy \\ &= \omega_{12} \wedge \theta_2 \\ d\theta_2 &= -\frac{1}{y^2}dy \wedge dy \\ &= 0 \\ &= -\omega_{12} \wedge \theta_1 \\ \omega_{12} &= \frac{1}{y}dx \\ d\omega_{12} &= -\frac{1}{y^2}dy \wedge dx \\ &= \frac{1}{y}dx \wedge \frac{1}{y}dy \\ &= \theta_1 \wedge \theta_2 \end{aligned}$$

So the Gaussian curvature of $\mathbb{H}^2$ is $K = -1$. The area of Poincare half-plane is then

$$\begin{aligned} \text{Area}(\mathbb{H}^2) &= \iint_{\mathbb{H}^2} \theta_1 \wedge \theta_2 \\ &= \iint_{\mathbb{H}^2} \frac{1}{y^2} dx \wedge dy \\ &= \iint_{\mathbb{H}^2} \frac{1}{y^2} dx\, dy \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \frac{1}{y^2} dy\, dx \\ &= \int_{-\infty}^{\infty} -\frac{1}{y}\bigg|_0^{\infty} dx \\ &= -\infty \end{aligned}$$

It should be noted that

$$\iint_A f\,dx \wedge dy = \iint_A f\,dx\,dy = \iint_A f\,dy\,dx = -\iint_A f\,dy \wedge dx \tag{73}$$

5.2. Covariant Derivative

DEFINITION. *The covariant derivative in $\mathbb{R}^3$ is a function:*

$$\begin{aligned} \nabla: \mathfrak{X}(\mathbb{R}^3) \times \mathfrak{X}(\mathbb{R}^3) &\rightarrow \mathfrak{X}(\mathbb{R}^3) \\ \nabla_V X &= \begin{pmatrix} V[X_1] \\ V[X_2] \\ V[X_3] \end{pmatrix} \end{aligned}$$

satisfying

1. $\mathbb{R}-\text{bilinear}$ *in both* V and X*;*
2. $f-\text{linear}$ *in* V*:* $\nabla_{fV} = f\nabla_V X$*;*
3. *Leibniz rule in* X*:* $\nabla_V(fX) = V[f]X + f\nabla_V X$.

DEFINITION. *On an open set U of a geometric surface, a function*

$$\nabla: \mathfrak{X}(U) \times \mathfrak{X}(U) \rightarrow \mathfrak{X}(U)$$

is a covariant derivative on U if it satisfies these 3 properties.

THEOREM. *The connection form* ω_{12} *on* (U, E_1, E_2) *defines a covariant derivative on U by*

$$\begin{aligned} \nabla_V E_1 &= \omega_{12}(V) E_2 \\ \nabla_V E_2 &= -\omega_{12}(V) E_1 \\ \nabla_V (f_1 E_1) &= V[f_1] E_1 + f_1 \omega_{12}(V) E_2 \\ \nabla_V (f_2 E_2) &= V[f_2] E_2 - f_2 \omega_{12}(V) E_1 \end{aligned}$$

An arbitrary $X = f_1 E_1 + f_2 E_2 \in \mathfrak{X}(U)$ *is*

$$\nabla_V(X) = (V[f_1] - f_2 \omega_{12}(V)) E_1 + (V[f_2] + f_1 \omega_{12}(V)) E_2$$

This $\nabla_V X$ *is clearly* $\mathbb{R}-\text{bilinear}$, $f-\text{linear}$ *in V, we can check it also satisfies the Leibniz rule.*

DEFINITION. *Let* α: $[a, b] \to M$ *be a curve in a geometric surface and let X be a vector field in M along the curve* α*, and* $\mathfrak{X}(\alpha^* T_p(M))$ *is a differentiable vector field in M along* α*. A covariant derivative along* α *is a function*

$$\frac{D}{dt} : \mathfrak{X}(\alpha^* T_p(M)) \to \mathfrak{X}(\alpha^* T_p(M))$$

such that

1. $\mathbb{R}-\text{linear}$ *in X;*
2. *Leibniz rule in X:* $\frac{D}{dt}(fX) = \frac{df}{dt} \cdot X + f \frac{DX}{dt}$*;*
3. *If X is the restriction of* $\tilde{X}$ *on M, then* $\frac{DX}{dt} = \nabla_{\alpha'(t)} \tilde{X}$*;*
4. $\frac{d}{dt} < V, W > = < \frac{DV}{dt}, W > + < V, \frac{DW}{dt} >$.

THEOREM. *Given a covariant derivative* ∇ *on M and a curve* $\alpha(t)$ *in M, there exists a unique covariant derivative* $\frac{D}{dt}$ *along* α.

Proof. On a framed open set (U, E_1, E_2),

$$\begin{aligned} V &= \sum V_i E_i \\ \frac{DV}{dt} &= \sum V_i'(t) E_i + \sum V_i \frac{DE_i}{dt} \\ &= \sum V_i'(t) E_i + \sum V_i \nabla_{\alpha'(t)} E_i \end{aligned}$$

Use this as the definition of $\frac{DV}{dt}$. Verify the 4 properties. □

DEFINITION. *If* α: $[a, b] \to M$ *is a curve on a geometric surface M, then* α' *is a vector field along* α.

$$\alpha'(t) \overset{\text{def}}{=\!=} \alpha_* \left(\frac{d}{dt} \right)$$

The acceleration is defined as

$$\alpha''(t) \overset{\text{def}}{=\!=} \frac{D}{dt} \alpha'(t)$$

A curve α *in a geometric surface is a geodesic if* $\alpha'' = 0$.

5.3. Gauss-Bonnet Theorem

Let β: $[a, b] \to M$ be a unit-speed curve in an oriented geometric surface, and $T = \beta' = \beta_* \left(\frac{d}{ds} \right)$. Because $\|T\| = 1$, $T' = \frac{DT}{ds}$ will be orthogonal to T,

$$\begin{aligned} < T(s), T(s) > &= \|T\|^2 = 1 \\ \frac{d}{ds} < T(s), T(s) > &= < \frac{DT}{ds}, T > + < T, \frac{DT}{ds} > = 0 \\ < \frac{DT}{ds}, T > &= 0 \end{aligned}$$

Since M is oriented, there is a positive oritented orthogonal frame T, N s.t. $T' = kN$ for some $k \in \mathbb{R}$. k is the geodesic curvature.

THEOREM. *A unit-speed curve on an oriented geometric surface is a geodesic iff* $k = 0$.

Suppose T makes an angle φ relative to E_1 in an oriented orthogonal frame E_1, E_2,

$$\begin{pmatrix} T \\ N \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

Then take the derivative

$$\begin{aligned} T' &= \frac{DT}{ds} \\ &= -\sin\varphi \frac{d\varphi}{ds} E_1 + \cos\varphi \frac{DE_2}{ds} + \cos\varphi \frac{d\varphi}{ds} E_2 + \sin\varphi \frac{DE_2}{ds} \\ &= \frac{d\varphi}{ds}(-\sin\varphi E_1 + \cos\varphi E_2) + \cos\varphi \nabla_{\beta'(s)} E_1 + \sin\varphi \nabla_{\beta'(s)} E_2 \\ &= \frac{d\varphi}{ds} N + (\cos\varphi)\omega_{12}(\beta') E_2 + (\sin\varphi)\omega_{21}(\beta') E_1 \\ &= \left(\frac{d\varphi}{ds} + \omega_{12}(\beta')\right) N \end{aligned}$$

Therefore the geodesic curvature is

$$k = \frac{d\varphi}{ds} + \omega_{12}(\beta') \tag{74}$$

DEFINITION. *The total geodesic curvature on* β *is defined as*

$$\begin{aligned} \int_a^b k(s) ds &= \int_a^b \frac{d\varphi}{ds} ds + \int_a^b \omega_{12}(\beta'(s)) ds \\ &= \varphi(b) - \varphi(a) + \int_\beta \omega_{12} \end{aligned} \tag{75}$$

THEOREM. (GAUSS-BONNET) *The total Gaussian curvature M of a compact orientable geometric surface M is 2π times its Euler characteristic:*

$$\iint_D K dM = 2\pi \mathcal{X}(M) \tag{76}$$

Proof. Let $\triangle_i =$ change of angle along ∂_i, ι_i is exterior angle and ε_i is exterior angle at the end of the ith edge. Total geodesic curvature on the boundary of a rectangle ∂D is

$$\begin{aligned} \sum_{i=1}^4 \int_{\partial_i} k &= \sum_{i=1}^4 \triangle_i + \sum_{i=1}^4 \int_{\partial_i} \omega_{12} \\ &= 2\pi - \sum_{i=1}^4 \varepsilon_i + \int_{\partial x} \omega_{12} \\ &= 2\pi - \sum_{i=1}^4 (\pi - \iota_i) + \iint_D d\omega_{12} \\ &= -2\pi + \sum_{i=1}^4 \iota_i - \iint_D K\theta_1 \wedge \theta_2 \\ &= -2\pi + \sum_{i=1}^4 \iota_i - \iint_D K dM \end{aligned}$$

Suppose M can be cut up into rectangle patches. Let v, e, f be the number of vertices, edges, and faces in a rectangle partition of M. Sum up total geodesic curvature,

$$\begin{aligned}
\sum_x \sum_{i=1}^4 \int_{\partial_i} k &= \sum_f -2\pi + \sum_v \iota_i - \iint_D K dM \\
0 &= -2\pi f + 2\pi v - \iint_D K dM \\
\iint_D K dM &= -4\pi f + 2\pi f + 2\pi v \\
&= -2\pi e + 2\pi f + 2\pi v \\
&= 2\pi(v - e + f) \\
&= 2\pi \mathcal{X}(M)
\end{aligned}$$

The theorem shows that total Gaussian curvature is a topological invariant. □

Theorem. *Let S be a surface, D is an oriented polygonal region in a geometric surface, k is the geodesic curvature, K is the Gaussian curvature at a point in D. If A_i is each angle of the irregular point. The Gauss-Bonnet Theorem is*

$$\sum_{i=1}^n (\pi - \iota_i) + \int_{\partial D} k ds + \iint_D K dM = 2\pi \mathcal{X}(M) \tag{77}$$

Proof. If we use a rectangle partition, and now the boundary curves survive,

$$\begin{aligned}
\sum_x \sum_{i=1}^4 \int_{\partial_i} k &= \sum_f -2\pi + \sum_v \iota_i - \iint_D K dM \\
\int_{\partial D} k ds &= -2\pi f + 2\pi(v - n) + \sum_{i=1}^n \iota_i - \iint_D K dM \\
\int_{\partial D} k ds + \iint_D K dM &= -4\pi f + 2\pi f + 2\pi v - 2n\pi + \sum_{i=1}^n \iota_i
\end{aligned}$$

Different from previously $4f = 2e$, here with the boundaries we have $4f = 2e - n$, thus

$$\begin{aligned}
\int_{\partial D} k ds + \iint_D K dM &= \pi(n - 2e) + 2\pi f + 2\pi v - 2n\pi + \sum_{i=1}^n \iota_i \\
\int_{\partial D} k ds + \iint_D K dM &= 2\pi(v - e + f) - n\pi + \sum_{i=1}^n \iota_i \\
\sum_{i=1}^n (\pi - \iota_i) + \int_{\partial D} k ds + \iint_D K dM &= 2\pi \mathcal{X}(M)
\end{aligned}$$

This is based on that the polygon can be partitioned by rectangles. □

Example. Geodesic triangle in Euclidean surface, where $k = 0, K = 0, \mathcal{X} = 1$, then

$$\begin{aligned}
\sum_{i=1}^3 (\pi - \iota_i) + \int_{\partial D} 0 ds + \iint_D 0 dM &= 2\pi \\
\sum_{i=1}^3 \iota_i &= \pi
\end{aligned}$$

More generally for a geodesic polygon in geodesic surface,

$$\begin{aligned}
\sum_{i=1}^n (\pi - \iota_i) + \int_{\partial D} 0 ds + \iint_D K dM &= 2\pi \\
\sum_{i=1}^n \iota_i &= (n-2)\pi + \iint_D K dM
\end{aligned}$$

Specifically for Euclidean space where $K=0$, $\sum_{i=1}^{n}\iota_i=(n-2)\pi$. If it's a geodesic triangle on a sphere with radius r, then we have $K=\frac{1}{r^2}$, $\mathcal{X}=1$, and

$$\begin{aligned}\sum_{i=1}^{3}(\pi-\iota_i)+\int_{\partial D}0ds+\iint_D\frac{1}{r^2}dM &= 2\pi\\ 3\pi-\sum_{i=1}^{3}\iota_i+\frac{\Delta}{r^2} &= 2\pi\\ \sum_{i=1}^{3}\iota_i &= \pi-\frac{\Delta}{r^2}\end{aligned}$$

COROLLARY. *Let M be a compact orientable surface. Then TFAE:*

1. *M has a continuous nowhere-vanish vector field V;*
2. *$\mathcal{X}(M)=1$;*
3. *M is a torus.*

Proof. Assume 1), let $E_1=\frac{V}{\|V\|}$, $E_2=J(E_1)$. So the entire surface is a framed open set. There is a unique connection form ω_{12} on M,

$$\begin{aligned}d\omega_{12} &= -K\theta_1\wedge\theta_2\\ &= -KdM\end{aligned}$$

According to Guass-Bonnet Theorem, then

$$0=\int_{\partial M}\omega_{12}=\int_M d\omega_{12}=-\iint_M KdM=-2\pi\mathcal{X}(M)$$

So $\mathcal{X}(M)=0$. Thus 1) $\Rightarrow$ 2), 2) $\Rightarrow$ 3) by Classification Theorem, 3) $\Rightarrow$ 1) by construction. □

6. MANIFOLDS

6.1. Topological Manifolds

DEFINITION. *A topological space M is locally Euclidean of dimension n if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. The pair $(U,\phi: U\to\mathbb{R}^n)$ is a chart, U is a coordinate neighborhood or a coordinate open set, and ϕ is a coordinate map or a coordinate system on U. A chart (U,ϕ) is centered at $p\in U$ if $\phi(p)=0$.*

DEFINITION. *A topological manifold is a Hausdorff, second countable, locally Euclidean space. It's said to be of dimension n if it's locally Euclidean of dimension n.*

DEFINITION. *Two charts $(U,\phi: U\to\mathbb{R}^n)$, $(V,\psi: V\to\mathbb{R}^n)$ of a topological manifold are $C^\infty-$*compatible *if the two maps*

$$\phi\circ\psi^{-1}: \psi(U\cap V)\to\phi(U\cap V),\quad \psi\circ\phi^{-1}: \phi(U\cap V)\to\psi(U\cap V)$$

are C^∞. These two maps are called the transition functions between the charts.

DEFINITION. *A C^∞ atlas or simply an atlas on a locally Euclidean space M is a collection $\mathfrak{U}=\{(U_\alpha,\phi_\alpha)\}$ of pairwise $C^\infty-$*compatible *charts that cover M, i.e., such that $M=\cup_\alpha U_\alpha$.*

An atlas $\mathfrak{M}$ on a locally Euclidean space is said to be maximal if it's not contained in a larger atlas; if $\mathfrak{U}$ is any other atlas containing $\mathfrak{M}$, then $\mathfrak{U}=\mathfrak{M}$.

DEFINITION. *A smooth or C^∞ manifold is a topological manifold M together with a maximal atlas. The maximal atlas is also called a differentiable structure on M. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is called a curve, a 2-dimensional manifold a surface, and an n-dimensional manifold an n-manifold.*

DEFINITION. *A Lie group is a C^∞ manifold G having a group structure s.t. the multiplication map*

$$\mu: G \times G \to G$$

and the inverse map

$$\iota: G \to G, \quad \iota(x) = x^{-1}$$

are both C^∞.

6.2. Categories and Functors

A category consists of a collection of elements, called objects, and for any two objects A and B, a set $\mathrm{Mor}(A, B)$ of elements, called morphisms from A to B, s.t. given any morphism $f \in \mathrm{Mor}(A, B)$ and any morphism $g \in \mathrm{Mor}(B, C)$, the composite $g \circ f \in \mathrm{Mor}(A, C)$ is defined. It satisfies:

i. the identity axiom: for each object A, there is an identity morphism $1_A \in \mathrm{Mor}(A, A)$ s.t. for any $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, A)$,

$$f \circ 1_A = f, \quad 1_A \circ g = g$$

ii. the associative axiom: for $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$, and $h \in \mathrm{Mor}(C, D)$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

If $f \in \mathrm{Mor}(A, B)$, we often write $f: A \to B$.

DEFINITION. *Two objects A and B in a category are said to be isomorphic if there are morphisms $f: A \to B$ and $g: B \to A$ s.t.*

$$g \circ f = 1_A, \quad f \circ g = 1_B$$

In this case both f and g are called isomorphisms.

DEFINITION. *A (covariant) functor $\mathcal{F}$ from one category $\mathcal{C}$ to another category $\mathcal{D}$ is a map that associates to each object A in $\mathcal{C}$ an object $\mathcal{F}(A)$ in $\mathcal{D}$ and to each morphism $f: A \to B$ there is a morphism $\mathcal{F}(f): \mathcal{F}(A) \to \mathcal{F}(B)$ s.t.*

i. $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$

ii. $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$

DEFINITION. *A contravariant functor $\mathcal{F}$ from one category $\mathcal{C}$ to another category $\mathcal{D}$ is a map that associates to each object A in $\mathcal{C}$ an object $\mathcal{F}(A)$ in $\mathcal{D}$ and to each morphism $f: A \to B$ there is a morphism $\mathcal{F}(f): \mathcal{F}(A) \to \mathcal{F}(B)$ s.t.*

i. $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$

ii. $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$

Example. The pushforward map $F_*: T_p(N) \to T_{F(p)}(M)$ is a functor because

$$(G \circ F)_* = G_* \circ F_*$$

The pullback map otherwises satisfies

$$(G \circ F)^* = F^* \circ G^*$$

6.3. Vector Bundle

A bundle map construction is a functor from the category of smooth manifolds to the category of vector bundles.

DEFINITION. *Let M be a smooth manifold, the tangent bundle of M is the union of all the tangent spaces of M:*

$$TM = \bigcup_{p \in M} T_p M = \coprod_{p \in M} T_p M$$

DEFINITION. *Product bundle is a special case of* $\pi: E \mapsto M$:

$$\pi: M \times V \mapsto M$$

7. APPENDIX

7.1. Generalization Map

The generalization from single-variable calculus to several-variable calculus is as follows [3].

$$\begin{array}{rcl} \text{indefinite integral} & \longrightarrow & \begin{cases} \text{solution to differential equations} \\ \text{integral of } a \text{ connection, vector field, or bundle} \end{cases} \\ \text{unsigned definite integral} & \longrightarrow & \text{Lebesgue integral} \longrightarrow \text{integration of } a \text{ measure space} \\ \text{signed definite integral} & \longrightarrow & \text{integration of forms} \end{array}$$

7.2. Notation Table

	Math		Physics
$\int_a^a f(x)dx = 0$	closed 1-form	$f(x)dx$	conservative force
	exact form		potential function

Table 2. Terminology Dictionary

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