Navigation

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1. GLOBAL POSITIONING SYSTEM

1.1. Accuracy

1.2. Integrity

Navigation system integrity [5] refers to the ability of the system to provide timely warnings to users when the system should not be used for navigation.

RAIM: receiver autonomous integrity monitoring.

1) Does a failure exist? 2) If so, which is the failed satellite?

Failure here is defined to mean that the solution of horizontal radial error is outside a specified limit, which is called "alarm limit".

1. Basic sanpshot RAIM schemes

$$y = G x_{\text{true}} + \epsilon \tag{1}$$

where y is the difference between the pseudorange and the predicted range;

x_{true} is true position deviation from the nominal position plus the user clock bias deviation;

ϵ is the measurement error vector.

Example 1. Range comparison method

Solve four equations; the resulting solution is then used to predict other measurements; compare them with the actual measured values for residuals. If residuals are small, then "no failure"; otherwise, "failure".

Example 2. Least-squares-residuals method

$$\hat{x}_{\text{LS}} = (G^T G)^{-1} G^T y \tag{2}$$

predict the six measurements

$$y_{\text{pred}} = G \hat{x}_{\text{LS}} \tag{3}$$

the residual is

$$w = y - y_{\text{pred}} = (I - G(G^T G)^{-1} G^T) y \tag{4}$$

the sum of squared errors is

$$\text{SSE} = w^T w \tag{5}$$

normalize SSE as the test statistic $\sqrt{\text{SSE}/(n-4)}$.

Example 3. Parity method

1.3. Continuity

1.4. Availability

2. INERTIAL NAVIGATION ALGORITHM

2.1. INS Mechanization

2.1.1. Golden rule

Velocity in e frame,

$$\left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_e = v_e \tag{6}$$

Golden rule,

$$\left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_a = \left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_b + \omega_{\mathrm{ab}} \times r \tag{7}$$

Acceleration measurement,

$$\left.\frac{d^2 r}{\mathrm{dt}^2}\right|_i = f + g \tag{8}$$

Acceleration in i frame,

$$\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i = \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times v_e \tag{9}$$

2.1.2. Velocity update

- Velocity in i Frame

$$\left.\frac{dr}{\mathrm{dt}}\right|_i = \left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times r = v_e + \omega_{\mathrm{ie}} \times r \tag{10}$$

Take the derivative on both sides,

$$\left.\frac{d^2 r}{\mathrm{dt}^2}\right|_i = \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i + \left.\frac{d(\omega_{\mathrm{ie}} \times r)}{\mathrm{dt}}\right|_i \tag{11}$$

Because

$$\begin{aligned} \left.\frac{d(\omega_{\mathrm{ie}} \times r)}{\mathrm{dt}}\right|_i &= \left.\frac{d\omega_{\mathrm{ie}}}{\mathrm{dt}}\right|_i \times r + \omega_{\mathrm{ie}} \times \left.\frac{\mathrm{dr}}{\mathrm{dt}}\right|_i \\ &= 0 \times r + \omega_{\mathrm{ie}} \times (v_e + \omega_{\mathrm{ie}} \times r) \\ &= \omega_{\mathrm{ie}} \times v_e + \omega_{\mathrm{ie}} \times (\omega_{\mathrm{ie}} \times r) \end{aligned} \tag{12}$$

Then

$$\begin{aligned} \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i &= \left.\frac{d^2 r}{\mathrm{dt}^2}\right|_i - \left.\frac{d(\omega_{\mathrm{ie}} \times r)}{\mathrm{dt}}\right|_i \\ &= f + g - (\omega_{\mathrm{ie}} \times v_e + \omega_{\mathrm{ie}} \times (\omega_{\mathrm{ie}} \times r)) \\ &= f - \omega_{\mathrm{ie}} \times v_e + (g - \omega_{\mathrm{ie}} \times (\omega_{\mathrm{ie}} \times r)) \\ &= f - \omega_{\mathrm{ie}} \times v_e + g_l \end{aligned} \tag{13}$$

Projected into i frame,

$$\begin{aligned} \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i^i &= f^i - \omega_{\mathrm{ie}}^i \times v_e^i + g_l^i \\ &= C_b^i f^b - \omega_{\mathrm{ie}}^i \times v_e^i + g_l^i \end{aligned} \tag{14}$$

- Velocity in e Frame

Using Golden rule,

$$\begin{aligned}
\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times v_e \\
f - \omega_{\mathrm{ie}} \times v_e + g_l &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e + \omega_{\mathrm{ie}} \times v_e \\
\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e &= f - 2\omega_{\mathrm{ie}} \times v_e + g_l \qquad (15)
\end{aligned}$$

Projected into e frame,

$$\begin{aligned}
\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e^e &= f^e - 2\omega_{\mathrm{ie}}^e \times v_e^e + g_l^e \\
&= C_b^e f^b - 2\omega_{\mathrm{ie}}^e \times v_e^e + g_l^e \qquad (16)
\end{aligned}$$

- Velocity in n Frame

Using Golden rule,

$$\begin{aligned}
\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_i &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_n + \omega_{\mathrm{in}} \times v_e \\
f - \omega_{\mathrm{ie}} \times v_e + g_l &= \left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_n + \omega_{\mathrm{in}} \times v_e \\
\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e &= f - \omega_{\mathrm{ie}} \times v_e + g_l - (\omega_{\mathrm{ie}} + \omega_{\mathrm{en}}) \times v_e \\
\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_e &= f - (2\omega_{\mathrm{ie}} + \omega_{\mathrm{en}}) \times v_e + g_l \qquad (17)
\end{aligned}$$

Projected into n frame,

$$\begin{aligned}
\left.\frac{\mathrm{dv}_e}{\mathrm{dt}}\right|_n^n &= f^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n + g_l^n \\
&= C_b^n f^b - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n + g_l^n \qquad (18)
\end{aligned}$$

Updated velocity

$$\begin{aligned}
v_e^n(t_k) &= v_e^n(t_{k-1}) + \int_{t_{k-1}}^{t_k} C_b^n(t) f^b(t) \mathrm{dt} + \int_{t_{k-1}}^{t_k} (g_l^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n) \mathrm{dt} \\
&\approx v_e^n(t_{k-1}) + \Delta v_f^n(t_k) + \Delta v_{g/\mathrm{cor}}^n(t_k) \qquad (19)
\end{aligned}$$

where

$$\begin{aligned}
\Delta v_f^n(t_k) &= \int_{t_{k-1}}^{t_k} C_b^n(t) f^b(t) \mathrm{dt} \\
&= \int_{t_{k-1}}^{t_k} C_{n(t_{k-1})}^{n(t)} C_{b(t_{k-1})}^{n(t_{k-1})} C_{b(t)}^{b(t_{k-1})} f^b(t) \mathrm{dt} \\
&= C_{n(t_{k-1})}^{n(t)} C_{b(t_{k-1})}^{n(t_{k-1})} \int_{t_{k-1}}^{t_k} C_{b(t)}^{b(t_{k-1})} f^b(t) \mathrm{dt} \\
&\approx (I - (0.5\boldsymbol{\zeta}_k \times)) C_{b(t_{k-1})}^{n(t_{k-1})} \Delta v_f^b(t_k) \qquad (20)
\end{aligned}$$

By definition,

$$\begin{aligned}
I - (0.5\boldsymbol{\zeta}_k \times) &= I - 0.5 \begin{pmatrix} 0 & -\boldsymbol{\zeta}_k[2] & \boldsymbol{\zeta}_k[1] \\ \boldsymbol{\zeta}_k[2] & 0 & -\boldsymbol{\zeta}_k[0] \\ -\boldsymbol{\zeta}_k[1] & \boldsymbol{\zeta}_k[0] & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0.5\boldsymbol{\zeta}_k[2] & -0.5\boldsymbol{\zeta}_k[1] \\ -0.5\boldsymbol{\zeta}_k[2] & 1 & 0.5\boldsymbol{\zeta}_k[0] \\ 0.5\boldsymbol{\zeta}_k[1] & -0.5\boldsymbol{\zeta}_k[0] & 1 \end{pmatrix} \qquad (21)
\end{aligned}$$

Also we have

$$
\begin{aligned}
\boldsymbol{\zeta}_k &= [\omega_{\text{ie}}^n + \omega_{\text{en}}^n]_{k-1/2}\Delta t_k & (22)\\
C_e^n &= R_y(-\varphi - \pi/2)R_z(\lambda)\\
&= \begin{pmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ -\sin\lambda & \cos\lambda & 0 \\ -\cos\varphi\cos\lambda & -\cos\varphi\sin\lambda & -\sin\varphi \end{pmatrix} & (23)\\
\omega_e &= 7.2921151467\times 10^{-5}\,\text{rad}/s & (24)\\
\omega_{\text{ie}}^e &= (\ 0\ \ 0\ \ \omega_e\ )^T & (25)\\
\omega_{\text{ie}}^n &= C_e^n \omega_{\text{ie}}^e\\
&= (\ \omega_e\cos\varphi\ \ 0\ \ -\omega_e\sin\varphi\ )^T & (26)\\
\omega_{\text{en}}^n &= \begin{pmatrix} \dot{\lambda}\cos\varphi \\ -\dot{\varphi} \\ -\dot{\lambda}\sin\varphi \end{pmatrix}\\
&= \begin{pmatrix} v_E/(R_N+h) \\ -v_N/(R_M+h) \\ -v_E\tan\varphi/(R_N+h) \end{pmatrix} & (27)\\
R_N &= \frac{a}{(1-e^2\sin^2\varphi)^{1/2}} & (28)\\
R_M &= \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}} & (29)\\
a &= 6378137.0 & (30)\\
f &= \frac{a-b}{a}\\
&= 1.0/298.257223563 & (31)
\end{aligned}
$$

Extraploating the position,

$$
\begin{aligned}
h_{k-1/2} &= h_{k-1} - \frac{v_D(t_{k-1})\Delta t_k}{2} & (32)\\
q_{n(k-1/2)}^{e(k-1)} &= q_{n(k-1)}^{e(k-1)} \star q_{n(k-1/2)}^{n(k-1)}\\
q_{n(k-1/2)}^{e(k-1/2)} &= q_{e(k-1)}^{e(k-1/2)} \star q_{n(k-1/2)}^{e(k-1)}\\
&= q_{e(k-1)}^{e(k-1/2)} \star \left(q_{n(k-1)}^{e(k-1)} \star q_{n(k-1/2)}^{n(k-1)} \right) & (33)
\end{aligned}
$$

q_n^e in terms of latitude, longitude, and altitude:

$$
q_n^e = \begin{pmatrix} \cos\left(-\frac{\pi}{4} - \frac{\varphi}{2}\right)\cos\left(\frac{\lambda}{2}\right) \\ -\sin\left(-\frac{\pi}{4} - \frac{\varphi}{2}\right)\sin\left(\frac{\lambda}{2}\right) \\ \sin\left(-\frac{\pi}{4} - \frac{\varphi}{2}\right)\cos\left(\frac{\lambda}{2}\right) \\ \cos\left(-\frac{\pi}{4} - \frac{\varphi}{2}\right)\sin\left(\frac{\lambda}{2}\right) \end{pmatrix} \tag{34}
$$

where longitude, λ, is ranged between$(-\pi, \pi]$, latitude, φ, is ranged between $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
when $\lambda = \pi$,

$$
q_n^e = \begin{pmatrix} 0 \\ -\sin\left(-\frac{\pi}{4} - \frac{\varphi}{2}\right) \\ 0 \\ \cos\left(-\frac{\pi}{4} - \frac{\varphi}{2}\right) \end{pmatrix} \tag{35}
$$

$$
\varphi = 2 * \left(-\frac{\pi}{4} - \arctan\left(-\frac{q_2}{q_4} \right) \right) \tag{36}
$$

when $\varphi = \frac{\pi}{2}$,

$$q_n^e = \begin{pmatrix} 0 \\ \sin\left(\frac{\lambda}{2}\right) \\ -\cos\left(\frac{\lambda}{2}\right) \\ 0 \end{pmatrix} \quad (37)$$

$$\lambda = 2*\arctan\left(-\frac{q_2}{q_3}\right) \quad (38)$$

otherwise,

$$\lambda = 2*\arctan\left(\frac{q_4}{q_1}\right) \quad (39)$$

$$\varphi = 2*\left(-\frac{\pi}{4} - \arctan\left(\frac{q_3}{q_1}\right)\right) \quad (40)$$

where

$$q_{n(k-1/2)}^{n(k-1)} = \begin{pmatrix} \cos\|0.5\boldsymbol{\zeta}_{k-1/2}\| \\ \frac{0.5\boldsymbol{\zeta}_{k-1/2}}{\|0.5\boldsymbol{\zeta}_{k-1/2}\|}\sin\|0.5\boldsymbol{\zeta}_{k-1/2}\| \end{pmatrix} \quad (41)$$

$$q_{e(k-1)}^{e(k-1/2)} = \begin{pmatrix} \cos\|0.5\xi_{k-1/2}\| \\ -\frac{0.5\xi_{k-1/2}}{\|0.5\xi_{k-1/2}\|}\sin\|0.5\xi_{k-1/2}\| \end{pmatrix} \quad (42)$$

$$\boldsymbol{\zeta}_{k-1/2} = \omega_{\mathrm{in}}^n(t_{k-1})\Delta t_k/2 \quad (43)$$

$$\xi_{k-1/2} = \omega_{\mathrm{ie}}^n \Delta t_k/2 \quad (44)$$

Extraploating the velocity,

$$\Delta v_e^n(t_{k-1}) = \Delta v_f^n(t_{k-1}) + \Delta v_{g/\mathrm{cor}}^n(t_{k-1}) \quad (45)$$

$$\begin{aligned} v_e^n(t_{k-1/2}) &= v_e^n(t_{k-1}) + \frac{1}{2}\Delta v_e^n(t_{k-1}) \\ &= v_e^n(t_{k-1}) + \frac{1}{2}(\Delta v_f^n(t_{k-1}) + \Delta v_{g/\mathrm{cor}}^n(t_{k-1})) \end{aligned} \quad (46)$$

Velocity correction of the gravity and coriolis terms,

$$\Delta v_{g/\mathrm{cor}}^n(t_k) = [g_l^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n]_{k-1/2}\Delta t_k \quad (47)$$

$$g_l^n = \begin{pmatrix} 0 & 0 & g \end{pmatrix}^T \quad (48)$$

$$g = g_0(1 + 5.27094 * 10^{-3}\sin^2\varphi + 2.32718 * 10^{-5}\sin^4\varphi) - 3.086 * 10^{-6}h \quad (49)$$

Since we have

$$C_{b(t)}^{b(t_{k-1})} \approx I + [\Delta\theta(t) \times] \quad (50)$$

$$\Delta\theta(t) = \int_{t_{k-1}}^{t} \omega_{\mathrm{ib}}^b(t)\mathrm{dt} \quad (51)$$

$$\Delta v(t) = \int_{t_{k-1}}^{t} f^b(t)\mathrm{dt} \quad (52)$$

$$\Delta\theta(t_{k-1}) = \Delta v(t_{k-1}) = 0 \quad (53)$$

where

$$\begin{aligned}
\Delta v_f^b(t_k) &= \int_{t_{k-1}}^{t_k} C_{b(t)}^{b(t_{k-1})} f^b(t)\mathrm{dt} \\
&\approx \int_{t_{k-1}}^{t_k} (I+[\Delta\theta(t)\times])f^b(t)\mathrm{dt} \\
&= \int_{t_{k-1}}^{t_k} f^b(t)\mathrm{dt} + \int_{t_{k-1}}^{t_k} (\Delta\theta(t)\times f^b(t))\mathrm{dt} \\
&= \Delta v(t_k) + \int_{t_{k-1}}^{t_k} (\Delta\theta(t)\times f^b(t))\mathrm{dt} \qquad (54)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\Delta\theta(t)\times f^b(t) &= \Delta\theta(t)\times\Delta\dot{v}(t) \\
&= \frac{d}{\mathrm{dt}}(\Delta\theta(t)\times\Delta v(t)) - \Delta\dot{\theta}(t)\times\Delta v(t) \\
&= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t)\times\Delta v(t)) + \frac{1}{2}(\Delta\dot{\theta}(t)\times\Delta v(t) + \Delta\theta(t)\times\Delta\dot{v}(t)) - \Delta\dot{\theta}(t)\times\Delta v(t) \\
&= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t)\times\Delta v(t)) + \frac{1}{2}(-\Delta\dot{\theta}(t)\times\Delta v(t) + \Delta\theta(t)\times\Delta\dot{v}(t)) \\
&= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t)\times\Delta v(t)) + \frac{1}{2}(\Delta v(t)\times\Delta\dot{\theta}(t) + \Delta\theta(t)\times\Delta\dot{v}(t)) \\
&= \frac{1}{2}\frac{d}{\mathrm{dt}}(\Delta\theta(t)\times\Delta v(t)) + \frac{1}{2}(\Delta v(t)\times\omega_{\mathrm{ib}}^b(t) + \Delta\theta(t)\times f^b(t)) \qquad (55)
\end{aligned}$$

Then

$$\begin{aligned}
\int_{t_{k-1}}^{t_k} (\Delta\theta(t)\times f^b(t))\mathrm{dt} &= \frac{1}{2}(\Delta\theta(t_k)\times\Delta v(t_k) - \Delta\theta(t_{k-1})\times\Delta v(t_{k-1})) + \\
&\quad \frac{1}{2}\int_{t_{k-1}}^{t_k} (\Delta v(t)\times\omega_{\mathrm{ib}}^b(t) + \Delta\theta(t)\times f^b(t))\mathrm{dt} \\
&= \frac{1}{2}(\Delta\theta(t_k)\times\Delta v(t_k)) + \\
&\quad \frac{1}{2}\int_{t_{k-1}}^{t_k} (\Delta v(t)\times\omega_{\mathrm{ib}}^b(t) + \Delta\theta(t)\times f^b(t))\mathrm{dt} \qquad (56)
\end{aligned}$$

Assuming the angular velocity and acceleration are linear during $t_{k-1}\sim t_k$ and $t_{k-2}\sim t_{k-1}$,

$$\omega_{\mathrm{ib}}^b(t) = a + 2b(t - t_{k-1}) \qquad (57)$$

$$f^b(t) = A + 2B(t - t_{k-1}) \qquad (58)$$

Using angular velocities and accelerations to resolve the coefficients,

$$\begin{aligned}
\Delta\theta(t_k) &= \int_{t_{k-1}}^{t_k} \omega_{\mathrm{ib}}^b(t)\mathrm{dt} \\
&= \int_{t_{k-1}}^{t_k} (a + 2b(t - t_{k-1}))\mathrm{dt} \qquad (59) \\
\Delta v(t_k) &= \int_{t_{k-1}}^{t_k} f^b(t)\mathrm{dt} \\
&= \int_{t_{k-1}}^{t_k} (A + 2B(t - t_{k-1}))\mathrm{dt} \qquad (60)
\end{aligned}$$

Plugging into the integral above,

$$\frac{1}{2}\int_{t_{k-1}}^{t_k} (\Delta v(t)\times\omega_{\mathrm{ib}}^b(t) + \Delta\theta(t)\times f^b(t))\mathrm{dt} = \frac{1}{12}(\Delta v(t_{k-1})\times\Delta\theta(t_k) + \Delta\theta(t_{k-1})\times\Delta v(t_k)) \qquad (61)$$

Summary of veloctiy update,

$$\begin{aligned}
v_e^n(t_k) &= v_e^n(t_{k-1}) + \int_{t_{k-1}}^{t_k} C_b^n(t) f^b(t)\mathrm{dt} + \int_{t_{k-1}}^{t_k} (g_l^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n)\mathrm{dt} \\
&= v_e^n(t_{k-1}) + \Delta v_f^n(t_k) + \Delta v_{g/\mathrm{cor}}^n(t_k) \\
&\approx v_e^n(t_{k-1}) + (I - (0.5\boldsymbol{\zeta}_k \times))C_{b(t_{k-1})}^{n(t_{k-1})} \Delta v_f^b(t_k) + \Delta v_{g/\mathrm{cor}}^n(t_k) \\
&\approx v_e^n(t_{k-1}) + (I - (0.5\boldsymbol{\zeta}_k \times))C_{b(t_{k-1})}^{n(t_{k-1})} \left(\Delta v(t_k) + \int_{t_{k-1}}^{t_k} (\Delta\theta(t) \times f^b(t))\mathrm{dt} \right) + \Delta v_{g/\mathrm{cor}}^n(t_k) \\
&= v_e^n(t_{k-1}) + (I - (0.5\boldsymbol{\zeta}_k \times))C_{b(t_{k-1})}^{n(t_{k-1})} \Bigg(\Delta v(t_k) + \Bigg(\frac{1}{2}(\Delta\theta(t_k) \times \Delta v(t_k)) + \frac{1}{2}\int_{t_{k-1}}^{t_k} (\Delta v(t) \times \\
&\quad \omega_{\mathrm{ib}}^b(t) + \Delta\theta(t) \times f^b(t))\mathrm{dt} \Bigg)\Bigg) + \Delta v_{g/\mathrm{cor}}^n(t_k) \\
&\approx v_e^n(t_{k-1}) + (I - (0.5\boldsymbol{\zeta}_k \times))C_{b(t_{k-1})}^{n(t_{k-1})} \Bigg(\Delta v(t_k) + \Bigg(\frac{1}{2}(\Delta\theta(t_k) \times \Delta v(t_k)) + \frac{1}{12}(\Delta v(t_{k-1}) \times \\
&\quad \Delta\theta(t_k) + \Delta\theta(t_{k-1}) \times \Delta v(t_k)) \Bigg)\Bigg) + [g_l^n - (2\omega_{\mathrm{ie}}^n + \omega_{\mathrm{en}}^n) \times v_e^n]_{k-1/2} \Delta t_k
\end{aligned} \tag{62}$$

2.1.3. Position update

Position in n frame,

$$\dot{r}^n = \begin{pmatrix} \frac{1}{R_M + h} & 0 & 0 \\ 0 & \frac{1}{(R_N + h)\cos\varphi} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_n \\ v_e \\ v_d \end{pmatrix} = D^{-1} v^n \tag{63}$$

$$r^n(t_{k+1}) = r^n(t_k) + \frac{1}{2} \begin{pmatrix} \frac{1}{R_M + h} & 0 & 0 \\ 0 & \frac{1}{(R_N + h)\cos\varphi} & 0 \\ 0 & 0 & -1 \end{pmatrix}_{k-\frac{1}{2}} (v_e^n(t_k) + v_e^n(t_{k+1}))\Delta t \tag{64}$$

2.1.4. Attitude update

Quaternion in terms of Euler angle (ZYX):

Rotating with Z axis, $\phi = \theta = 0$, quaternion representation: $q_\psi = \cos\frac{\psi}{2} - \mathrm{k}\sin\frac{\psi}{2}$

Rotating with Y axis, $\phi = \psi = 0$, quaternion representation: $q_\theta = \cos\frac{\theta}{2} - \mathrm{j}\sin\frac{\theta}{2}$

Rotating with X axis, $\psi = \theta = 0$, quaternion representation: $q_\phi = \cos\frac{\phi}{2} - \mathrm{i}\sin\frac{\phi}{2}$

The whole ratation is then, $q = q_\phi q_\theta q_\psi$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} - \sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} \end{pmatrix} \tag{65}$$

DCM in terms of Euler angle

$$C_b^n = \begin{pmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{pmatrix} \tag{66}$$

DCM in terms of quaternion

$$C_B^A = \begin{pmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2(q_2q_3-q_1q_4) & 2(q_2q_4+q_1q_3) \\ 2(q_2q_3+q_1q_4) & q_1^2-q_2^2+q_3^2-q_4^2 & 2(q_3q_4-q_1q_2) \\ 2(q_2q_4-q_1q_3) & 2(q_3q_4+q_1q_2) & q_1^2-q_2^2-q_3^2+q_4^2 \end{pmatrix} \tag{67}$$

Euler angle in terms of DCM

$$\begin{aligned} \theta &= \tan^{-1}\frac{\sin\theta}{\cos\theta} \\ &= \tan^{-1}\frac{-c_{31}}{\sqrt{c_{32}^2+c_{33}^2}} \end{aligned} \tag{68}$$

$$\begin{aligned} \phi &= \tan^{-1}\frac{\sin\phi}{\cos\phi} \\ &= \tan^{-1}\frac{c_{32}}{c_{33}} \end{aligned} \tag{69}$$

$$\begin{aligned} \psi &= \tan^{-1}\frac{\sin\psi}{\cos\psi} \\ &= \tan^{-1}\frac{c_{21}}{c_{11}} \end{aligned} \tag{70}$$

The approximation of DCM with small angle

$$C_\beta^\alpha = \begin{pmatrix} 1 & \psi_{\beta\alpha} & -\theta_{\beta\alpha} \\ -\psi_{\beta\alpha} & 1 & \phi_{\beta\alpha} \\ \theta_{\beta\alpha} & -\phi_{\beta\alpha} & 1 \end{pmatrix} = I_3 - (\triangle\boldsymbol{\Theta}\times) \tag{71}$$

Gyroscope output,

$$\triangle\theta_{b(t_{k-1})b(t_k)} = \int_{t_{k-1}}^{t_k} \omega_{\mathrm{ib}}^b(t)\mathrm{d}t \tag{72}$$

$$C_{b(t_{k-1})}^{b(t_k)} = I - (\triangle\theta_{b(t_{k-1})b(t_k)}\times) \tag{73}$$

$$C_{b(t_k)}^{b(t_{k-1})} = I + (\triangle\theta_{b(t_{k-1})b(t_k)}\times) \tag{74}$$

The update of q_b^n,

$$\begin{aligned} q_{b(k)}^{n(k-1)} &= q_{b(k-1)}^{n(k-1)} * q_{b(k)}^{b(k-1)} \\ q_{b(k)}^{n(k)} &= q_{n(k-1)}^{n(k)} * q_{b(k)}^{n(k-1)} \\ &= q_{n(k-1)}^{n(k)} * \left(q_{b(k-1)}^{n(k-1)} * q_{b(k)}^{b(k-1)}\right) \end{aligned} \tag{75}$$

$$q_{b(k)}^{b(k-1)} = \begin{pmatrix} \cos\|0.5\phi_k\| \\ \frac{0.5\phi_k}{\|0.5\phi_k\|}\sin\|0.5\phi_k\| \end{pmatrix} \tag{76}$$

$$\begin{aligned} \dot{\phi} &\approx w_{\mathrm{ib}}^b + \frac{1}{2}\phi\times w_{\mathrm{ib}}^b + \frac{1}{12}\phi\times(\phi\times w_{\mathrm{ib}}^b) \\ &\approx w_{\mathrm{ib}}^b + \frac{1}{2}\triangle\theta(t)\times w_{\mathrm{ib}}^b \end{aligned} \tag{77}$$

where

$$\triangle\theta(t) = \int_{t_{k-1}}^{t} \omega_{\mathrm{ib}}^b(t)\mathrm{d}t \tag{78}$$

Accordingly,

$$\begin{aligned} \phi_k &= \int_{t_{k-1}}^{t_k}\left[\omega_{\mathrm{ib}}^b(t) + \frac{1}{2}\triangle\theta(t)\times w_{\mathrm{ib}}^b\right]\mathrm{d}t \\ &\approx \triangle\theta_k + \frac{1}{12}\triangle\theta_{k-1}\times\triangle\theta_k \end{aligned} \tag{79}$$

Also we have

$$q_{n(k-1)}^{n(k)} = \begin{pmatrix} \cos\|0.5\zeta_k\| \\ -\frac{0.5\zeta_k}{\|0.5\zeta_k\|}\sin\|0.5\zeta_k\| \end{pmatrix} \tag{80}$$

To calculate ζ_k, recompute q_n^e at $t_{k-1/2}$,

$$q_{\delta\theta} = \left(q_{n(k-1)}^{e(k-1)}\right)^{-1} * q_{n(k)}^{e(k)} \tag{81}$$

$$q_{n(k-1/2)}^{e(k-1/2)} = q_{n(k-1)}^{e(k-1)} * q_{0.5\delta\theta} \tag{82}$$

Axis-angle representation in terms of quaternion,

$$q_b^a = (\ q_1 \ \ q_2 \ \ q_3 \ \ q_4\ )^T \tag{83}$$

$$\|0.5\phi\| = \tan^{-1}\frac{\sin\|0.5\phi\|}{\cos\|0.5\phi\|} = \tan^{-1}\frac{\sqrt{q_2^2+q_3^2+q_4^2}}{q_1} \tag{84}$$

$$\begin{aligned} f &\equiv \frac{\sin\|0.5\phi\|}{\|\phi\|} \\ &= 0.5 * \frac{\sin\|0.5\phi\|}{\|0.5\phi\|} \\ &= \frac{1}{2}\left(1 - \frac{\|0.5\phi\|^2}{3!} + \frac{\|0.5\phi\|^4}{5!} - \frac{\|0.5\phi\|^6}{7!} + \ldots\right) \end{aligned} \tag{85}$$

$$\phi = \frac{1}{f}(\ q_2 \ \ q_3 \ \ q_4\ )^T \tag{86}$$

when $q_1 = 0$,

$$\phi = \pi(\ q_2 \ \ q_3 \ \ q_4\ )^T \tag{87}$$

3. ADVANCED INS

3.1. IMU Data Format

3.1.1. Coordinate frames

Body frame (b-frame)
  Practical inertial frame (i-frame)
  Earth fixed frame (ECEF, e-frame)
  Local level frame (LLF, n-frame)

3.1.2. Measurements

$$\triangle\theta = \int_{t_{k-1}}^{t_k} \omega_{ib}^b(t)dt \tag{88}$$

$$\triangle v = \int_{t_{k-1}}^{t_k} f^b(t)dt \tag{89}$$

3.1.3. Parameters to resolve

$$\boldsymbol{v_e^n} \tag{90}$$

3.2. Velocity Update

3.2.1. First time-derivative of a vector in a reference frame

$$\left.\frac{dr}{dt}\right|_a = \left.\frac{dr}{dt}\right|_b + \omega_{ab} \times r \tag{91}$$

3.2.2. Second time-derivative of a vector in a reference frame

$$\begin{aligned}
\left.\frac{d^2r}{dt^2}\right|_a &= \left.\frac{d}{dt}\right|_a\left(\left.\frac{dr}{dt}\right|_b + \omega_{ab}\times r\right)\\
&= \left.\frac{d}{dt}\right|_a\left(\left.\frac{dr}{dt}\right|_b\right) + \left.\frac{d\omega_{ab}}{dt}\right|_a \times r + \omega_{ab}\times\left.\frac{dr}{dt}\right|_a\\
&= \left(\left.\frac{d}{dt}\right|_b\left(\left.\frac{dr}{dt}\right|_b\right) + \omega_{ab}\times\left.\frac{dr}{dt}\right|_b\right) + \left.\frac{d\omega_{ab}}{dt}\right|_a \times r + \omega_{ab}\times\left(\left.\frac{dr}{dt}\right|_b + \omega_{ab}\times r\right)\\
&= \left.\frac{d^2r}{dt^2}\right|_b + 2\omega_{ab}\times\left.\frac{dr}{dt}\right|_b + \left.\frac{d\omega_{ab}}{dt}\right|_a \times r + \omega_{ab}\times(\omega_{ab}\times r) \qquad (92)
\end{aligned}$$

the 2nd time-derivative of any vector in a reference frame A can be calculated as

$$\begin{aligned}
\left.\frac{d^2r}{dt^2}\right|_a &= \left.\frac{d^2r}{dt^2}\right|_b + \left.\frac{d\omega_{ab}}{dt}\right|_a \times r + \omega_{ab}\times(\omega_{ab}\times r) + 2\omega_{ab}\times\left.\frac{dr}{dt}\right|_b\\
\text{relative:} &\quad \left.\frac{d^2r}{dt^2}\right|_b\\
\text{tangential:} &\quad \left.\frac{d\omega_{ab}}{dt}\right|_a \times r\\
\text{centripetal:} &\quad \omega_{ab}\times(\omega_{ab}\times r)\\
\text{Coriolis:} &\quad 2\omega_{ab}\times\left.\frac{dr}{dt}\right|_b
\end{aligned}$$

Coriolis Wiki

Dynamics book: 7.4.4

3.2.3. Velocity update

velocity equation in e frame

$$\begin{aligned}
\left.\frac{d^2r}{dt^2}\right|_i &= \left.\frac{d^2r}{dt^2}\right|_e + \left.\frac{d\omega_{ie}}{dt}\right|_i \times r + \omega_{ie}\times(\omega_{ie}\times r) + 2\omega_{ie}\times\left.\frac{dr}{dt}\right|_e\\
f+g &= \left.\frac{dv_e}{dt}\right|_e + 0 + \omega_{ie}\times(\omega_{ie}\times r) + 2\omega_{ie}\times v_e\\
\left.\frac{dv_e}{dt}\right|_e &= f + (g - \omega_{ie}\times(\omega_{ie}\times r)) - 2\omega_{ie}\times v_e\\
\left.\frac{dv_e}{dt}\right|_e &= f + g_l - 2\omega_{ie}\times v_e \qquad (93)
\end{aligned}$$

velocity equation in n frame

$$\begin{aligned}
\left.\frac{dv_e}{dt}\right|_e &= \left.\frac{dv_e}{dt}\right|_n + \omega_{en}\times v_e\\
f + g_l - 2\omega_{ie}\times v_e &= \left.\frac{dv_e}{dt}\right|_n + \omega_{en}\times v_e\\
\left.\frac{dv_e}{dt}\right|_n &= f + g_l - (2\omega_{ie} + w_{en})\times v_e \qquad (94)
\end{aligned}$$

3.3. Attitude Update

3.3.1. Special Orthogonal Matrices

$$\mathrm{SO}(n) = \{C \in \mathbb{R}^{n\times n} | C^TC = CC^T = I, \det C = 1\} \qquad (95)$$

3.3.2. Angular velocity and orthogonal basis vectors

$$\begin{aligned}
\left.\frac{db_x}{dt}\right|_a &= \left.\frac{db_x}{dt}\right|_b + \omega_{ab} \times b_x \\
&= 0 + \omega_{ab} \times b_x \\
&= \omega_{ab} \times b_x
\end{aligned} \tag{96}$$

$$\begin{aligned}
\left.\frac{db_y}{dt}\right|_a &= \left.\frac{db_y}{dt}\right|_b + \omega_{ab} \times b_y \\
&= \omega_{ab} \times b_y
\end{aligned} \tag{97}$$

$$\begin{aligned}
\left.\frac{db_z}{dt}\right|_a &= \left.\frac{db_z}{dt}\right|_b + \omega_{ab} \times b_z \\
&= \omega_{ab} \times b_z
\end{aligned} \tag{98}$$

using scalar triple product equation

$$a \cdot (b \times c) = (a \times b) \cdot c \tag{99}$$

we have

$$\begin{aligned}
\left.\frac{db_x}{dt}\right|_a \cdot b_y &= (\omega_{ab} \times b_x) \cdot b_y \\
&= \omega_{ab} \cdot (b_x \times b_y) \\
&= \omega_{ab} \cdot b_z
\end{aligned} \tag{100}$$

$$\left.\frac{db_y}{dt}\right|_a \cdot b_z = \omega_{ab} \cdot b_x \tag{101}$$

$$\left.\frac{db_z}{dt}\right|_a \cdot b_x = \omega_{ab} \cdot b_y \tag{102}$$

we also have

$$\begin{aligned}
\left.\frac{db_x}{dt}\right|_a \cdot b_z &= (\omega_{ab} \times b_x) \cdot b_z \\
&= \omega_{ab} \cdot (b_x \times b_z) \\
&= -\omega_{ab} \cdot b_y
\end{aligned} \tag{103}$$

$$\left.\frac{db_y}{dt}\right|_a \cdot b_x = -\omega_{ab} \cdot b_z \tag{104}$$

$$\left.\frac{db_z}{dt}\right|_a \cdot b_y = -\omega_{ab} \cdot b_x \tag{105}$$

therefore

$$\begin{aligned}
\omega_{ab} &= (\omega_{ab} \cdot b_x) b_x + (\omega_{ab} \cdot b_y) b_y + (\omega_{ab} \cdot b_z) b_z \\
&= \left(\left.\frac{db_y}{dt}\right|_a \cdot b_z\right) b_x + \left(\left.\frac{db_z}{dt}\right|_a \cdot b_x\right) b_y + \left(\left.\frac{db_x}{dt}\right|_a \cdot b_y\right) b_z \\
&= -\left(\left.\frac{db_z}{dt}\right|_a \cdot b_y\right) b_x - \left(\left.\frac{db_x}{dt}\right|_a \cdot b_z\right) b_y - \left(\left.\frac{db_y}{dt}\right|_a \cdot b_x\right) b_z
\end{aligned} \tag{106}$$

3.3.3. Rotation matrices and angular velocity

$$C_b^a = \begin{pmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{pmatrix} \tag{107}$$

By definition,

$$\begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & C_{zz} \end{pmatrix}^T \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \tag{108}$$

the angular velocity ω_{ab} in terms of C_{ij} is

$$\begin{aligned} \omega_{ab} &= \left(\frac{db_y}{dt}\bigg|_a \cdot b_z\right) b_x + \left(\frac{db_z}{dt}\bigg|_a \cdot b_x\right) b_y + \left(\frac{db_x}{dt}\bigg|_a \cdot b_y\right) b_z \\ &= (C_{xz}\dot{C_{xy}} + C_{yz}\dot{C_{yy}} + C_{zz}\dot{C_{zy}})b_x + (C_{xx}\dot{C_{xz}} + C_{yx}\dot{C_{yz}} + C_{zx}\dot{C_{zz}})b_y + (C_{xy}\dot{C_{xx}} + C_{yy}\dot{C_{yx}} + \\ &\quad C_{zy}\dot{C_{zx}})b_z \end{aligned} \tag{109}$$

$$\begin{aligned} \omega_{ab} &= -\left(\frac{db_z}{dt}\bigg|_a \cdot b_y\right) b_x - \left(\frac{db_x}{dt}\bigg|_a \cdot b_z\right) b_y - \left(\frac{db_y}{dt}\bigg|_a \cdot b_x\right) b_z \\ &= -(C_{xy}\dot{C_{xz}} + C_{yy}\dot{C_{yz}} + C_{zy}\dot{C_{zz}})b_x - (C_{xz}\dot{C_{xx}} + C_{yz}\dot{C_{yx}} + C_{zz}\dot{C_{zx}})b_y - (C_{xx}\dot{C_{xy}} + C_{yx}\dot{C_{yy}} + \\ &\quad C_{zx}\dot{C_{zy}})b_z \end{aligned} \tag{110}$$

because

$$\frac{db_x}{dt}\bigg|_a \cdot b_x = C_{xx}\dot{C_{xx}} + C_{yx}\dot{C_{yx}} + C_{zx}\dot{C_{zx}} \tag{111}$$

$$\begin{aligned} \frac{db_x}{dt}\bigg|_a \cdot b_x &= (\omega_{ab} \times b_x) \cdot b_x \\ &= \omega_{ab} \cdot (b_x \times b_x) \\ &= 0 \end{aligned} \tag{112}$$

thus

$$C_{xx}\dot{C_{xx}} + C_{yx}\dot{C_{yx}} + C_{zx}\dot{C_{zx}} = 0 \tag{113}$$

$$C_{xy}\dot{C_{xy}} + C_{yy}\dot{C_{yy}} + C_{zy}\dot{C_{zy}} = 0 \tag{114}$$

$$C_{xz}\dot{C_{xz}} + C_{yz}\dot{C_{yz}} + C_{zz}\dot{C_{zz}} = 0 \tag{115}$$

the end result is

$$\begin{aligned} \omega_{ab}\times &= \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} C_{xx} & C_{yx} & C_{zx} \\ C_{xy} & C_{yy} & C_{zy} \\ C_{xz} & C_{yz} & C_{zz} \end{pmatrix} \begin{pmatrix} \dot{C_{xx}} & \dot{C_{xy}} & \dot{C_{xz}} \\ \dot{C_{yx}} & \dot{C_{yy}} & \dot{C_{yz}} \\ \dot{C_{zx}} & \dot{C_{zy}} & \dot{C_{zz}} \end{pmatrix} \end{aligned} \tag{116}$$

$$\omega_{ab}\times = (C_b^a)^T \dot{C_b^a} \tag{117}$$

$$(C_b^a)^{-1} = (C_b^a)^T \tag{118}$$

$$\omega_{ab}\times = (C_b^a)^{-1} \dot{C_b^a} \tag{119}$$

$$\dot{C_b^a} = C_b^a(\omega_{ab}\times)$$

Dynamics book: 9.4.1

3.3.4. Poisson's kinematical differential equations

$$\dot{C}_b^a = C_b^a(\omega_{ab}\times) \tag{120}$$

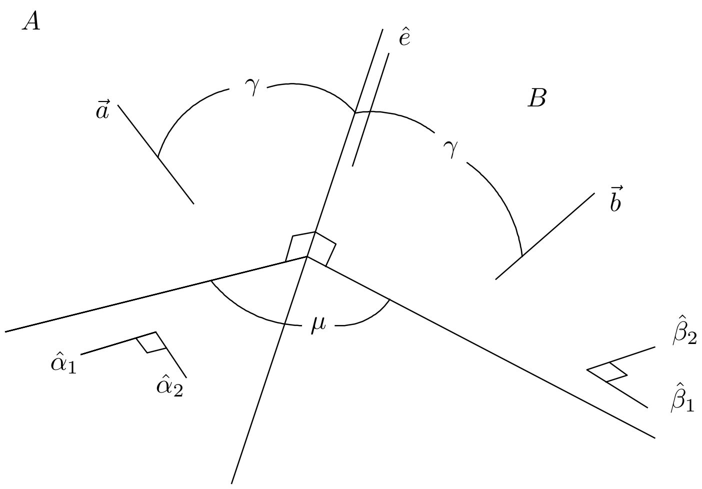
3.3.5. Axis-angle representation

$$C = C_b^a(\omega_{ab}\times) \tag{121}$$

3.3.6. Rodrigues' rotation formula

$$b = acos(\mu)+(e\times a)sin(\mu)+(1-cos(\mu))(e\cdot a)e \tag{122}$$

Proof.



$$b = cos(\gamma)e+sin(\gamma)\beta_1$$

because

$$\beta_1 = cos(\mu)\alpha_1+sin(\mu)\alpha_2$$

then

$$b = cos(\gamma)e+cos(\mu)sin(\gamma)\alpha_1+sin(\mu)sin(\gamma)\alpha_2$$

we also have

$$\begin{aligned} a &= cos(\gamma)e+sin(\gamma)\alpha_1 \\ e\times a &= sin(\gamma)\alpha_2 \\ e\cdot a = a\cdot e &= cos(\gamma) \end{aligned}$$

substitute them back,

$$\begin{aligned} b &= cos(\gamma)e+cos(\mu)sin(\gamma)\alpha_1+sin(\mu)sin(\gamma)\alpha_2 \\ &= cos(\gamma)e+cos(\mu)(a-cos(\gamma)e)+sin(\mu)(e\times a) \\ &= cos(\mu)a+(1-cos(\mu))cos(\gamma)e+sin(\mu)(e\times a) \\ &= acos(\mu)+(e\times a)sin(\mu)+(1-cos(\mu))(e\cdot a)e \end{aligned}$$

□

3.3.7. Angular velocity in terms of $\mu\hat{e}$

$$\begin{aligned}\omega_{ab} &= sin(\mu)\frac{de}{dt}\Big|_b + \dot{\mu}e + (cos(\mu)-1)e \times \frac{de}{dt}\Big|_b \\ &= sin(\mu)\frac{de}{dt}\Big|_a + \dot{\mu}e + (cos(\mu)-1)e \times \frac{de}{dt}\Big|_a \end{aligned} \tag{123}$$

Proof.

$$\begin{aligned}\omega_{ab} &= \omega_{a\alpha} + \omega_{\alpha\beta} + \omega_{\beta b} \\ &= -\dot{\gamma}\alpha_2 + \dot{\mu}e + \dot{\gamma}\beta_2\end{aligned}$$

using rotation matrix substitute

$$\alpha_2 = sin(\mu)\beta_1 + cos(\mu)\beta_2$$

and rearranging, yields

$$\begin{aligned}\omega_{ab} &= -\dot{\gamma}\alpha_2 + \dot{\mu}e + \dot{\gamma}\beta_2 \\ &= -\dot{\gamma}(sin(\mu)\beta_1 + cos(\mu)\beta_2) + \dot{\mu}e + \dot{\gamma}\beta_2 \\ &= -sin(\mu)\dot{\gamma}\beta_1 + \dot{\mu}e + (1-cos(\mu))\dot{\gamma}\beta_2\end{aligned}$$

because

$$\begin{aligned}\frac{de}{dt}\Big|_b &= \frac{de}{dt}\Big|_\beta + \omega_{b\beta} \times e \\ &= 0 + (-\dot{\gamma}\beta_2) \times e \\ &= -\dot{\gamma}\beta_1 \\ e \times \frac{de}{dt}\Big|_b &= e \times (-\dot{\gamma}\beta_1) \\ &= -\dot{\gamma}\beta_2\end{aligned}$$

Similarly we get

$$\begin{aligned}-\dot{\gamma}\alpha_1 &= \frac{de}{dt}\Big|_a \\ -\dot{\gamma}\alpha_2 &= e \times \frac{de}{dt}\Big|_a\end{aligned}$$

then

$$\begin{aligned}\omega_{ab} &= sin(\mu)\frac{de}{dt}\Big|_b + \dot{\mu}e + (cos(\mu)-1)e \times \frac{de}{dt}\Big|_b \\ &= sin(\mu)\frac{de}{dt}\Big|_a + \dot{\mu}e + (cos(\mu)-1)e \times \frac{de}{dt}\Big|_a\end{aligned}$$

□

Example 4. Simple angular velocity, which means $\hat{e}$ is fixed in both A and B,

$$\begin{aligned}\frac{de}{dt}\Big|_a &= 0 \\ \frac{de}{dt}\Big|_b &= 0\end{aligned}$$

then

$$\omega_{ab} = \dot{\mu}e$$

3.3.8. Kinematical differential equations for $\mu\hat{e}$

Reference: Advanced Dynamics 9.5.2

$$
\begin{aligned}
e \cdot e &= 1 \\
e \cdot \frac{de}{dt} &= \frac{1}{2}\frac{d(e \cdot e)}{dt} \\
&= 0 \\
e \times \frac{de}{dt} \cdot e &= 0
\end{aligned}
$$

because $a \times (b \times c) \; = \; b(a \cdot c) - c(a \cdot b)$, then

$$
\begin{aligned}
e \times \left(e \times \frac{de}{dt}\right) &= e\left(e \cdot \frac{de}{dt}\right) - \frac{de}{dt}(e \cdot e) \\
&= e(0) - \frac{de}{dt}(1) \\
&= -\frac{de}{dt}
\end{aligned}
$$

Vector equations can be written in matrix form as

$$
\begin{aligned}
\omega_{ab} \cdot e &= sin(\mu)\frac{de}{dt} \cdot e\Big|_b + \dot{\mu}e \cdot e + (cos(\mu) - 1)e \times \frac{de}{dt} \cdot e\Big|_b \\
&= \dot{\mu}
\end{aligned}
$$

$$
\begin{pmatrix} \omega_{ab} - \dot{\mu}e \\ e \times \omega_{ab} \end{pmatrix} = \begin{pmatrix} sin(\mu) & cos(\mu) - 1 \\ 1 - cos(\mu) & sin(\mu) \end{pmatrix} \begin{pmatrix} \frac{de}{dt}\Big|_b \\ e \times \frac{de}{dt}\Big|_b \end{pmatrix} \tag{124}
$$

Solving these two equations using $tan\left(\frac{\mu}{2}\right) = \frac{1 - cos(\mu)}{sin(\mu)}$,

$$
\frac{de}{dt}\Big|_b = \frac{1}{2}\left(\frac{cos\left(\frac{\mu}{2}\right)}{sin\left(\frac{\mu}{2}\right)}(\omega_{ab} - \dot{\mu}e) + e \times \omega_{ab}\right) \tag{125}
$$

3.3.9. Unit quaternion/axis-angle representation

Define

$$
\begin{aligned}
q_r &= \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \\
\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} &= \begin{pmatrix} cos\left(\frac{\mu}{2}\right) \\ e_1 sin\left(\frac{\mu}{2}\right) \\ e_2 sin\left(\frac{\mu}{2}\right) \\ e_3 sin\left(\frac{\mu}{2}\right) \end{pmatrix} \\
\phi &= \mu\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}
\end{aligned}
$$

Kinematical differential equations for $\mu\hat{e}$ indicates,

$$
sin\left(\frac{\mu}{2}\right)\frac{de}{dt}\Big|_b = \frac{1}{2}\left(cos\left(\frac{\mu}{2}\right)(\omega_{ab} - \dot{\mu}e) + sin\left(\frac{\mu}{2}\right)e \times \omega_{ab}\right)
$$

By definition,

$$
\begin{aligned}
q_0 &= cos\Big(\frac{\mu}{2}\Big) \\
\dot{q}_0 &= -\frac{1}{2}sin\Big(\frac{\mu}{2}\Big)\dot{\mu} \\
&= -\frac{1}{2}sin\Big(\frac{\mu}{2}\Big)(\omega_{ab}\cdot e) \\
&= -\frac{1}{2}\omega_{ab}\cdot q_r
\end{aligned}
$$

$$
\begin{aligned}
q_r &= sin\Big(\frac{\mu}{2}\Big)e \\
\frac{dq_r}{dt}\Big|_b &= sin\Big(\frac{\mu}{2}\Big)\frac{de}{dt}\Big|_b + \frac{1}{2}cos\Big(\frac{\mu}{2}\Big)\dot{\mu}e \\
&= \frac{1}{2}\Big(cos\Big(\frac{\mu}{2}\Big)\omega_{ab} + sin\Big(\frac{\mu}{2}\Big)e\times\omega_{ab}\Big) \\
&= \frac{1}{2}(q_0\omega_{ab} + q_r\times\omega_{ab})
\end{aligned}
$$

$$
\begin{aligned}
\dot{q}_0 &= -\frac{1}{2}(\omega_{ab}\cdot q_r) \\
&= \frac{1}{2}\begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dq_r}{dt}\Big|_b &= \frac{1}{2}(q_0\omega_{ab} + q_r\times\omega_{ab}) \\
&= \frac{1}{2}(q_0\omega_{ab} - (\omega_{ab}\times)q_r) \\
&= \frac{1}{2}\left(q_0\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} - \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right) \\
&= \frac{1}{2}\begin{pmatrix} \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}
\end{aligned}
$$

$$
\begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{126}
$$

3.4. Tensor Analysis

3.4.1. Rodrigues' rotation formula in dyadic form

The vector form:

$$
b = acos(\mu) + (e\times a)sin(\mu) + (1-cos(\mu))(e\cdot a)e
$$

The dyadic form:

$$b = Ra \tag{127}$$

where

$$R = Icos(\mu) + sin(\mu)\Omega + (1 - cos(\mu))\omega\omega \tag{128}$$
$$\Omega = \omega_x(kj - jk) + \omega_y(ik - ki) + \omega_z(ji - ij) \tag{129}$$

The effect of Ω on a is the cross product

$$\Omega \cdot a = \omega \times a$$

Since $\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = C_b^a \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$, then $\begin{pmatrix} a_x & a_y & a_z \end{pmatrix} = \begin{pmatrix} b_x & b_y & b_z \end{pmatrix} C_a^b$. The rotation of dyadics is

$$\begin{aligned} \Omega &= \begin{pmatrix} a_x & a_y & a_z \end{pmatrix} \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \\ &= \begin{pmatrix} b_x & b_y & b_z \end{pmatrix} C_a^b \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix} C_b^a \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \end{aligned}$$

thus

$$\Omega^b = C_a^b \Omega^a C_b^a \tag{130}$$

Dyadic Wiki

3.4.2. Pauli matrices

A set of three 2×2 Hermitian and unitary complex matrices.

$$\delta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\delta_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\delta_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3.4.3. Quaternion group

$$\begin{aligned} H &= \{\pm\mathbf{1}, \pm\boldsymbol{i}, \pm\boldsymbol{j}, \pm\boldsymbol{k}\} \\ &= \{\pm I, \pm i\delta_3, \pm i\delta_2, \pm i\delta_1\} \\ \mathbf{1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \boldsymbol{i} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ \boldsymbol{j} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \boldsymbol{k} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{aligned}$$

then

$$\begin{aligned}\boldsymbol{i}^2=\boldsymbol{j}^2=\boldsymbol{k}^2 &= -\mathbf{1}\\ \boldsymbol{ij}=-\boldsymbol{ji} &= \boldsymbol{k}\\ \boldsymbol{jk}=-\boldsymbol{kj} &= \boldsymbol{i}\\ \boldsymbol{ki}=-\boldsymbol{ik} &= \boldsymbol{j}\end{aligned}$$

3.5. Finite Angles and Rotations

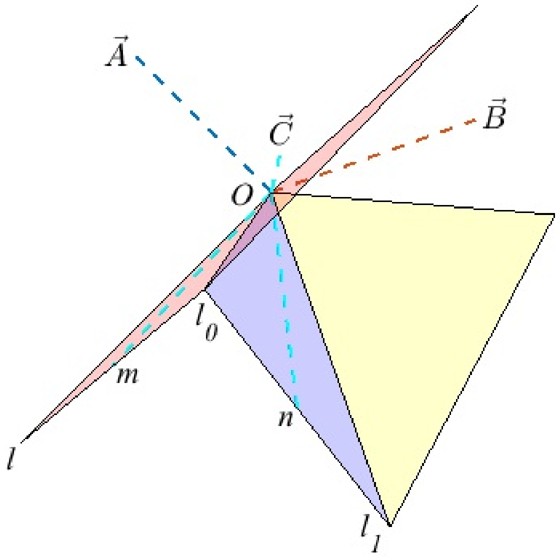


Figure 1. Rotations about two intersecting axes

Definition. *The angle between two intersecting lines, B and C, may be represented by [3]*

$$\vec{A}_{(l-l_0)}=\left(\vec{1}_l\times\vec{1}_{l_0}\right)q_{A_{(l-l_0)}},\quad \text{where } q_{A_{(l-l_0)}}=\frac{A_{(l-l_0)}}{\sin(A_{(l-l_0)})} \tag{131}$$

$A_{(l-l_0)}$ *is the radian measure of the angle from l to l_0.*

3.5.1. Arbitrary rotations about any two intersecting axes

Lines m and n bisect the respective angles A and B. Thus

$$\begin{aligned}\vec{A} &= 2\vec{A}_{(m-l_0)}\\ \vec{B} &= 2\vec{B}_{(l_0-n)}\end{aligned}$$

The rotation $2\vec{A}_{(m-l_0)}$ is equivalent to two successive 180° rotations about m and l_0; similarly, $2\vec{B}_{(l_0-n)}$ is equivalent to two successive 180° rotations about l_0 and n. Since the two intermediate rotations about l_0 cancel, the result is equivalent to two successive 180° rotations about m and n, which is represented by $2\vec{A}_{(m-n)}$. It has been shown that

$$\begin{aligned}\vec{A}\#\vec{B} &= 2\vec{A}_{(m-n)}\\ &= 2\left(\vec{A}_{(m-l_0)}(+)\vec{B}_{(l_0-n)}\right)\\ &= 2\left(\frac{\vec{A}}{2}(+)\frac{\vec{B}}{2}\right) \qquad (132)\end{aligned}$$

(132) expresses the definition of the operation # in terms of the previously defined operation (+). It also proves that two successive rotations about intersecting axes is itself a rotation, which is a theorem of Euler. (+) is defined as follows

$$\vec{A}_{\angle mOl_0}(+)\vec{B}_{\angle l_0On} = \vec{C}_{\angle mOn} \tag{133}$$

Both of (+) and # satisfy the associativity

$$\begin{aligned}\vec{A}(+)[\vec{B}(+)\vec{C}] &= [\vec{A}(+)\vec{B}](+)\vec{C}\\ \vec{A}\#[\vec{B}\#\vec{C}] &= [\vec{A}\#\vec{B}]\#\vec{C}\end{aligned}$$

If $\vec{A}$ and $\vec{B}$ are parallel vectors, which means the angles are on the same plane, it's true that

$$\vec{A}(+)\vec{B} = \vec{A}\#\vec{B} = \vec{A} + \vec{B}$$

Two operators $T_{(2)A}$ and T_A are defined by

$$\begin{aligned}T_{(2)A}(\vec{B}) &= (-\vec{A})(+)\vec{B}(+)\vec{A}\\ T_A(\vec{B}) &= (-\vec{A})\#\vec{B}\#\vec{A}\end{aligned}$$

The above operators follow

$$\begin{aligned}T_{(2)A}(\vec{B}) &= T_{2A}(\vec{B})\\ (-\vec{A})(+)\vec{B}(+)\vec{A} &= (-2\vec{A})\#\vec{B}\#(2\vec{A})\end{aligned} \tag{134}$$

3.5.2. Derivatives of operation in (+)

Let $\vec{A}_t$ denote the rotation vector at time t, and $\vec{A}_{t+\triangle t}$ the vector at time $t+\triangle t$, then the derivative is given by

$$\frac{d\vec{A}}{dt} = \lim_{\triangle t\to 0}\frac{1}{\triangle t}(\vec{A}_{t+\triangle t} - \vec{A}_t)$$

For operator (+), the derivatives are

$$\begin{aligned}\frac{D_R\vec{A}}{Dt} &= \lim_{\triangle t\to 0}\frac{1}{\triangle t}((-\vec{A}_t)(+)\vec{A}_{t+\triangle t})\\ &= \frac{d\vec{A}}{dt} + \frac{\vec{A}}{q_A}\times\left(\frac{1}{q_A}\frac{d\vec{A}}{dt}\right) + \frac{1}{A^2}\left(1-\frac{\cos(A)}{q_A}\right)\vec{A}\times\left(\vec{A}\times\frac{d\vec{A}}{dt}\right)\\ &= \frac{1}{2\cos(A)}\left\{\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{q_A}\right) + T_{2A}\left[\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{q_A}\right)\right]\right\}\end{aligned}$$

$$\begin{aligned}\frac{D_L\vec{A}}{Dt} &= \lim_{\triangle t\to 0}\frac{1}{\triangle t}(\vec{A}_{t+\triangle t}(+)(-\vec{A}_t))\\ &= \frac{d\vec{A}}{dt} - \frac{\vec{A}}{q_A}\times\left(\frac{1}{q_A}\frac{d\vec{A}}{dt}\right) + \frac{1}{A^2}\left(1-\frac{\cos(A)}{q_A}\right)\vec{A}\times\left(\vec{A}\times\frac{d\vec{A}}{dt}\right)\\ &= \frac{1}{2\cos(A)}\left\{\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{q_A}\right) + T_{(-2A)}\left[\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{q_A}\right)\right]\right\}\end{aligned}$$

Inversely we have

$$\begin{aligned}\frac{d\vec{A}}{dt} &= \frac{D_R\vec{A}}{Dt} - \left(\vec{A}\times\frac{D_R\vec{A}}{Dt}\right) + \frac{1-q_A\cos(A)}{A^2}\vec{A}\times\left(\vec{A}\times\frac{D_R\vec{A}}{Dt}\right)\\ &= \frac{D_L\vec{A}}{Dt} + \left(\vec{A}\times\frac{D_L\vec{A}}{Dt}\right) + \frac{1-q_A\cos(A)}{A^2}\vec{A}\times\left(\vec{A}\times\frac{D_L\vec{A}}{Dt}\right)\end{aligned}$$

It can be further proved that

$$\begin{aligned} T_{(-2A)}\left[\frac{D_R\vec{A}}{Dt}\right] &= \frac{D_L\vec{A}}{Dt} \\ T_{(2A)}\left[\frac{D_L\vec{A}}{Dt}\right] &= \frac{D_R\vec{A}}{Dt} \end{aligned}$$

Thus we can define a third type of derivative, which is symmetric in form.

$$\frac{D_S\vec{A}}{Dt} = T_{(-A)}\left[\frac{D_R\vec{A}}{Dt}\right] = T_{(A)}\left[\frac{D_L\vec{A}}{Dt}\right] = \frac{1}{2\cos(A)}\left\{T_A\left[\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{q_A}\right)\right] + T_{(-A)}\left[\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{q_A}\right)\right]\right\}$$

It follows that

$$\begin{aligned} \frac{D_S\vec{A}}{Dt} &= \frac{d\vec{A}}{dt} + \frac{1}{A^2}\left(1 - \frac{1}{q_A}\right)\vec{A} \times \left(\vec{A} \times \frac{d\vec{A}}{dt}\right) \\ \frac{d\vec{A}}{dt} &= \frac{D_S\vec{A}}{Dt} + \frac{1-q_A}{A^2}\vec{A} \times \left(\vec{A} \times \frac{D_S\vec{A}}{Dt}\right) \end{aligned}$$

3.5.3. Derivatives of operation in (#)

Similarly for operator #, we have

$$\begin{aligned} \frac{D_R^*\vec{A}}{Dt} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}((-\vec{A}_t)\#\vec{A}_{t+\Delta t}) \\ &= 2\frac{D_R}{Dt}\left(\frac{\vec{A}}{2}\right) \\ &= \frac{d\vec{A}}{dt} + \frac{1-\cos(A)}{A^2}\left(\vec{A} \times \frac{d\vec{A}}{dt}\right) + \frac{1}{A^2}\left(1 - \frac{1}{q_A}\right)\vec{A} \times \left(\vec{A} \times \frac{d\vec{A}}{dt}\right) \\ &= \frac{1}{\cos\left(\frac{A}{2}\right)}\left\{\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{2q_{\frac{A}{2}}}\right) + T_A\left[\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{2q_{\frac{A}{2}}}\right)\right]\right\} \end{aligned}$$

$$\begin{aligned} \frac{D_L^*\vec{A}}{Dt} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}(\vec{A}_{t+\Delta t}\#(-\vec{A}_t)) \\ &= 2\frac{D_L}{Dt}\left(\frac{\vec{A}}{2}\right) \\ &= \frac{d\vec{A}}{dt} - \frac{1-\cos(A)}{A^2}\left(\vec{A} \times \frac{d\vec{A}}{dt}\right) + \frac{1}{A^2}\left(1 - \frac{1}{q_A}\right)\vec{A} \times \left(\vec{A} \times \frac{d\vec{A}}{dt}\right) \\ &= \frac{1}{\cos\left(\frac{A}{2}\right)}\left\{\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{2q_{\frac{A}{2}}}\right) + T_{(-A)}\left[\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{2q_{\frac{A}{2}}}\right)\right]\right\} \end{aligned}$$

Inversely we have

$$\begin{aligned} \frac{d\vec{A}}{dt} &= \frac{D_R^*\vec{A}}{Dt} - \frac{1}{2}\left(\vec{A} \times \frac{D_R^*\vec{A}}{Dt}\right) + \frac{1}{A^2}\left(1 - \frac{A\sin(A)}{2(1-\cos(A))}\right)\vec{A} \times \left(\vec{A} \times \frac{D_R^*\vec{A}}{Dt}\right) \\ &= \frac{D_L^*\vec{A}}{Dt} + \frac{1}{2}\left(\vec{A} \times \frac{D_L^*\vec{A}}{Dt}\right) + \frac{1}{A^2}\left(1 - \frac{A\sin(A)}{2(1-\cos(A))}\right)\vec{A} \times \left(\vec{A} \times \frac{D_L^*\vec{A}}{Dt}\right) \end{aligned} \tag{135}$$

Thses relations show clearly that

$$T_{(-A)}\left[\frac{D_R^*\vec{A}}{Dt}\right] = \frac{D_L^*\vec{A}}{Dt}$$
$$T_{(A)}\left[\frac{D_L^*\vec{A}}{Dt}\right] = \frac{D_R^*\vec{A}}{Dt}$$

Thus we can define the symmetric derivative as

$$\frac{D_S^*\vec{A}}{Dt} = T_{\left(-\frac{A}{2}\right)}\left[\frac{D_R^*\vec{A}}{Dt}\right] = T_{\left(\frac{A}{2}\right)}\left[\frac{D_L^*\vec{A}}{Dt}\right] = \frac{1}{\cos\left(\frac{A}{2}\right)}\left\{T_{\left(\frac{A}{2}\right)}\left[\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{2q_{\frac{A}{2}}}\right)\right] + T_{\left(-\frac{A}{2}\right)}\left[\frac{d}{\mathrm{dt}}\left(\frac{\vec{A}}{2q_{\frac{A}{2}}}\right)\right]\right\}$$

It follows that

$$\frac{D_S^*\vec{A}}{Dt} = \frac{d\vec{A}}{dt} + \frac{1}{A^2}\left(1 - \frac{1}{q_{\frac{A}{2}}}\right)\vec{A}\times\left(\vec{A}\times\frac{d\vec{A}}{dt}\right)$$
$$\frac{d\vec{A}}{dt} = \frac{D_S^*\vec{A}}{Dt} + \frac{1 - q_{\frac{A}{2}}}{A^2}\vec{A}\times\left(\vec{A}\times\frac{D_S^*\vec{A}}{Dt}\right)$$

3.5.4. Angular velocities of spaces

Let S_r denote a reference space, S_1 and S_2 denote two spaces in a state of motion with respect to S_r. The rotation carrying S_r into S_1 is $\vec{A}_{r1}$, which satisfies

$$\left(\vec{A}_{r1}\right)_t \#(\vec{\omega}_{r1}dt) = \left(\vec{A}_{r1}\right)_{t+dt} = \left(\vec{A}_{r1}\right)_t \#\left(D_R^*\vec{A}_{r1}\right)_r$$

where the subscript r indicates the change as noted by an observer at rest with respect to S_r.

$$\vec{\omega}_{r1} = \left(\frac{D_R^*\vec{A}_{r1}}{Dt}\right)_r, \quad \vec{\omega}_{r2} = \left(\frac{D_R^*\vec{A}_{r2}}{Dt}\right)_r, \quad \vec{\omega}_{12} = \left(\frac{D_R^*\vec{A}_{12}}{Dt}\right)_1$$

For inertial reference space S_r, the rotation from S_1 to S_2 is constant in time

$$\left(\vec{A}_{12}\right)_{r(t+dt)} = \left(\vec{A}_{12}\right)_r$$

For space S_1, we have

$$\left(\vec{A}_{12}\right)_{1(t+dt)} = T_{\left(D_R^*\vec{A}_{r1}\right)}\left[\left(\vec{A}_{12}\right)_r\right] \tag{136}$$

By definition, the time variance can be calculated in both S_r and S_1 as follows.

$$\left(\vec{A}_{12}\right)_{t+dt} = \left(\vec{A}_{12}\right)_{r(t+dt)} \#\left(D_R^*\vec{A}_{12}\right)_r = \left(\vec{A}_{12}\right)_{1(t+dt)} \#\left(D_R^*\vec{A}_{12}\right)_1 \tag{137}$$

It should be noted that the subscript 12 here means rotating spaces, instead of rotating lines.

$$\left(\vec{A}_{r1}\right)_{t+dt} \#\left(\vec{A}_{12}\right)_{t+dt} = \left(\vec{A}_{r2}\right)_{t+dt}$$
$$\left(\vec{A}_{12}\right)_{t+dt} = \left(-\vec{A}_{r1}\right)_{t+dt} \#\left(\vec{A}_{r2}\right)_{t+dt}$$
$$= \left(-\left(\left(\vec{A}_{r1}\right)_t \#\left(D_R^*\vec{A}_{r1}\right)_r\right)\right) \#\left(\vec{A}_{r2}\right)_t \#\left(D_R^*\vec{A}_{r2}\right)_r$$
$$= \left(-D_R^*\vec{A}_{r1}\right)_r \#\left(-\vec{A}_{r1}\right)_t \#\left(\vec{A}_{r2}\right)_t \#\left(D_R^*\vec{A}_{r2}\right)_r$$

From (137) and (136) we have

$$\begin{aligned}
\left(D_R^*\vec{A}_{12}\right)_1 &= \left(-\left(\vec{A}_{12}\right)_{1(t+dt)}\right)\#\left(\vec{A}_{12}\right)_{t+dt}\\
&= \left(-\left(\vec{A}_{12}\right)_{1(t+dt)}\right)\#\left(-D_R^*\vec{A}_{r1}\right)_r\#\left(-\vec{A}_{r1}\right)_t\#\left(\vec{A}_{r2}\right)_t\#\left(D_R^*\vec{A}_{r2}\right)_r\\
&= \left(T_{(D_R^*\vec{A}_{r1})}\left[\left(\vec{A}_{12}\right)_r\right]\right)\#\left(-D_R^*\vec{A}_{r1}\right)_r\#\left(-\vec{A}_{r1}\right)_t\#\left(\vec{A}_{r2}\right)_t\#\left(D_R^*\vec{A}_{r2}\right)_r\\
&= \left(\left(-D_R^*\vec{A}_{r1}\right)_r\#\left(-\vec{A}_{12}\right)_r\#\left(D_R^*\vec{A}_{r1}\right)_r\right)\#\left(-D_R^*\vec{A}_{r1}\right)_r\#\left(-\vec{A}_{r1}\right)_t\#\left(\vec{A}_{r2}\right)_t\#\left(D_R^*\vec{A}_{r2}\right)_r\\
&= \left(-D_R^*\vec{A}_{r1}\right)_r\#\left(-\vec{A}_{12}\right)_r\#\left(-\vec{A}_{r1}\right)_t\#\left(\vec{A}_{r2}\right)_t\#\left(D_R^*\vec{A}_{r2}\right)_r\\
&= \left(-D_R^*\vec{A}_{r1}\right)_r\#\left(-\vec{A}_{12}\right)_r\#\left(\vec{A}_{12}\right)_{r(t)}\#\left(D_R^*\vec{A}_{r2}\right)_r\\
&= \left(-D_R^*\vec{A}_{r1}\right)_r\#\left(D_R^*\vec{A}_{r2}\right)_r
\end{aligned}$$

The following expression is thus obtained

$$\left(\frac{D_R^*\vec{A}_{12}}{Dt}\right)_1 = \left(\frac{D_R^*\vec{A}_{r2}}{Dt}\right)_r - \left(\frac{D_R^*\vec{A}_{r1}}{Dt}\right)_r$$

In other words,

$$\begin{aligned}
\vec{\omega}_{12} &= \vec{\omega}_{r2} - \vec{\omega}_{r1}\\
\vec{\omega}_{r2} &= \vec{\omega}_{r1} + \vec{\omega}_{12}
\end{aligned}$$

3.6. Bortz Equation

3.6.1. Numerical approximation

Example. The attitude update equation for inertial navigation algorithm is

$$q_{b(k)}^{n(k)} = q_{n(k-1)}^{n(k)} * \left(q_{b(k-1)}^{n(k-1)} * q_{b(k)}^{b(k-1)}\right) \tag{138}$$

in (138) the last term is

$$q_{b(k)}^{b(k-1)} = \begin{pmatrix} \cos\|0.5\phi_k\| \\ \frac{0.5\phi_k}{\|0.5\phi_k\|}\sin\|0.5\phi_k\| \end{pmatrix} \tag{139}$$

use Bortz equation to calculate ϕ,

$$\dot{\phi} = \omega_{\mathrm{ib}}^b + \frac{1}{2}\phi \times \omega_{\mathrm{ib}}^b + \frac{1}{\|\phi\|^2}\left(1 - \frac{\|\phi\|\sin(\|\phi\|)}{2(1-\cos(\|\phi\|))}\right)\phi \times (\phi \times \omega_{\mathrm{ib}}^b) \tag{140}$$

since $tan\left(\frac{\theta}{2}\right) = \frac{1-cos(\theta)}{sin(\theta)}$, simplify (140) into

$$\dot{\phi} = \omega_{\mathrm{ib}}^b + \frac{1}{2}\phi \times \omega_{\mathrm{ib}}^b + \frac{1}{\|\phi\|^2}\left(1 - \frac{\|\phi\|\cos\left(\frac{\|\phi\|}{2}\right)}{2\sin\left(\frac{\|\phi\|}{2}\right)}\right)\phi \times (\phi \times \omega_{\mathrm{ib}}^b) \tag{141}$$

use Taylor expanation at $x = 0$,

$$\frac{1}{\|\phi\|^2}\left(1 - \frac{\|\phi\|\cos\left(\frac{\|\phi\|}{2}\right)}{2\sin\left(\frac{\|\phi\|}{2}\right)}\right) = \frac{1}{12} + \frac{\|\phi\|^2}{720} + \frac{\|\phi\|^4}{30240} + O(\|\phi\|^5) \tag{142}$$

under small angle approximation, (140) turns into

$$\dot{\phi} \approx \omega_{\mathrm{ib}}^b + \frac{1}{2}\phi \times \omega_{\mathrm{ib}}^b + \frac{1}{12}\phi \times (\phi \times \omega_{\mathrm{ib}}^b) \tag{143}$$

further approximation produces

$$\begin{aligned}\dot{\phi} &\approx \omega_{\mathrm{ib}}^b + \left(\frac{1}{2}\phi \times \omega_{\mathrm{ib}}^b + \frac{1}{12}\phi \times (\phi \times \omega_{\mathrm{ib}}^b)\right) \\ &\approx \omega_{\mathrm{ib}}^b + \frac{1}{2}\triangle\theta \times \omega_{\mathrm{ib}}^b \end{aligned} \tag{144}$$

integrate (143) from t_k to t_{k+1},

$$\begin{aligned}\int_{t_{k-1}}^{t_k} \dot{\phi} dt &\approx \int_{t_{k-1}}^{t_k} \left(\omega_{\mathrm{ib}}^b + \frac{1}{2}\triangle\theta \times \omega_{\mathrm{ib}}^b\right) dt \\ \phi_k &\approx \int_{t_{k-1}}^{t_k} \omega_{\mathrm{ib}}^b \mathrm{dt} + \frac{1}{2}\int_{t_{k-1}}^{t_k} (\triangle\theta \times \omega_{\mathrm{ib}}^b)\mathrm{dt} \\ &= \triangle\theta_k + \frac{1}{2}\int_{t_{k-1}}^{t_k} (\triangle\theta \times \omega_{\mathrm{ib}}^b)\mathrm{dt}\end{aligned} \tag{145}$$

assuming linear variation of w_{ib}^b between $t_{k-2} \sim t_k$,

$$w_{\mathrm{ib}}^b \approx a + 2b(t - t_{k-1}) \tag{146}$$

also by definition

$$\begin{aligned}\triangle\theta &= \int_{t_{k-1}}^{t} \omega_{\mathrm{ib}}^b \mathrm{dt} \\ &\approx a(t - t_{k-1}) + b((t - t_{k-1})^2 - (t_{k-1} - t_{k-1})^2) \\ &= a(t - t_{k-1}) + b(t - t_{k-1})^2\end{aligned} \tag{147}$$

use measurements $\triangle\theta_{k-1}$ and $\triangle\theta_k$ to calculate the coefficients a and b,

$$\begin{aligned}\triangle\theta_k &= \int_{t_{k-1}}^{t_k} \omega_{\mathrm{ib}}^b dt \\ &= \int_{t_{k-1}}^{t_k} (a + 2b(t - t_{k-1}))dt \\ &= a(t_k - t_{k-1}) + b((t_k - t_{k-1})^2 - (t_{k-1} - t_{k-1})^2) \\ &= \triangle t\, a + (\triangle t)^2\, b\end{aligned} \tag{148}$$

$$\begin{aligned}\triangle\theta_{k-1} &= \int_{t_{k-2}}^{t_{k-1}} \omega_{\mathrm{ib}}^b dt \\ &= \int_{t_{k-2}}^{t_{k-1}} (a + 2b(t - t_{k-1}))dt \\ &= a(t_{k-1} - t_{k-2}) + b((t_{k-1} - t_{k-1})^2 - (t_{k-2} - t_{k-1})^2) \\ &= \triangle t\, a - (\triangle t)^2\, b\end{aligned} \tag{149}$$

a and b are solved to be

$$a = \frac{\triangle\theta_k + \triangle\theta_{k-1}}{2\triangle t} \tag{150}$$

$$b = \frac{\triangle\theta_k - \triangle\theta_{k-1}}{2(\triangle t)^2} \tag{151}$$

which means

$$\begin{aligned}w_{\mathrm{ib}}^b &\approx a + 2b(t - t_{k-1}) \\ &= \frac{\triangle\theta_k + \triangle\theta_{k-1}}{2\triangle t} + \frac{\triangle\theta_k - \triangle\theta_{k-1}}{(\triangle t)^2}(t - t_{k-1})\end{aligned} \tag{152}$$

by definition

$$\begin{aligned}
\triangle\theta &= \int_{t_{k-1}}^{t} \omega_{\mathrm{ib}}^{b}\mathrm{dt} \\
&\approx a(t-t_{k-1})+b(t-t_{k-1})^2
\end{aligned}$$

$$\begin{aligned}
\triangle\theta\times\omega_{\mathrm{ib}}^{b} &= (a(t-t_{k-1})+b(t-t_{k-1})^2)\times(a+2b(t-t_{k-1})) \\
&= (t-t_{k-1})(a\times a)+2(a\times b)(t-t_{k-1})^2+(b\times a)(t-t_{k-1})^2+2(b\times b)(t-t_{k-1})^3 \\
&= 2(a\times b)(t-t_{k-1})^2+(b\times a)(t-t_{k-1})^2
\end{aligned}$$

$$\begin{aligned}
\int_{t_{k-1}}^{t_k}(\triangle\theta\times\omega_{\mathrm{ib}}^{b})\mathrm{dt} &= \frac{2}{3}(a\times b)(t_k-t_{k-1})^3+\frac{1}{3}(b\times a)(t_k-t_{k-1})^3 \\
&= \frac{2}{3}(a\times b)(\triangle t)^3+\frac{1}{3}(b\times a)(\triangle t)^3 \\
&= \frac{2(\triangle t)^3}{3}\left(\frac{\triangle\theta_k+\triangle\theta_{k-1}}{2\triangle t}\right)\times\left(\frac{\triangle\theta_k-\triangle\theta_{k-1}}{2(\triangle t)^2}\right)+\frac{(\triangle t)^3}{3}\left(\frac{\triangle\theta_k-\triangle\theta_{k-1}}{2(\triangle t)^2}\right)\times \\
&\quad \left(\frac{\triangle\theta_k+\triangle\theta_{k-1}}{2\triangle t}\right) \\
&= \frac{1}{6}(\triangle\theta_k+\triangle\theta_{k-1})\times(\triangle\theta_k-\triangle\theta_{k-1})+\frac{1}{12}(\triangle\theta_k-\triangle\theta_{k-1})\times(\triangle\theta_k+\triangle\theta_{k-1}) \\
&= \frac{1}{6}(\triangle\theta_k\times\triangle\theta_k-\triangle\theta_k\times\triangle\theta_{k-1}+\triangle\theta_{k-1}\times\triangle\theta_k-\triangle\theta_{k-1}\times\triangle\theta_{k-1})+ \\
&\quad \frac{1}{12}(\triangle\theta_k\times\triangle\theta_k+\triangle\theta_k\times\triangle\theta_{k-1}-\triangle\theta_{k-1}\times\triangle\theta_k-\triangle\theta_{k-1}\times\triangle\theta_{k-1}) \\
&= \frac{1}{6}(-\triangle\theta_k\times\triangle\theta_{k-1}+\triangle\theta_{k-1}\times\triangle\theta_k)+\frac{1}{12}(\triangle\theta_k\times\triangle\theta_{k-1}-\triangle\theta_{k-1}\times\triangle\theta_k) \\
&= -\frac{1}{12}\triangle\theta_k\times\triangle\theta_{k-1}+\frac{1}{12}\triangle\theta_{k-1}\times\triangle\theta_k
\end{aligned}$$

because for vectors, $a\times b=-b\times a$, then

$$\begin{aligned}
\int_{t_{k-1}}^{t_k}(\triangle\theta\times\omega_{\mathrm{ib}}^{b})\mathrm{dt} &= -\frac{1}{12}\triangle\theta_k\times\triangle\theta_{k-1}+\frac{1}{12}\triangle\theta_{k-1}\times\triangle\theta_k \\
&= \frac{1}{6}\triangle\theta_{k-1}\times\triangle\theta_k \qquad (153)
\end{aligned}$$

according to (145),

$$\begin{aligned}
\phi_k &\approx \triangle\theta_k+\frac{1}{2}\int_{t_{k-1}}^{t_k}(\triangle\theta\times\omega_{\mathrm{ib}}^{b})\mathrm{dt} \\
&\approx \triangle\theta_k+\frac{1}{2}\left(\frac{1}{6}\triangle\theta_{k-1}\times\triangle\theta_k\right) \\
&= \triangle\theta_k+\frac{1}{12}\triangle\theta_{k-1}\times\triangle\theta_k \qquad (154)
\end{aligned}$$

3.6.2. Coning motion

DEFINITION. *Coning motion is defined as the condition whereby an angular rate vector is itself rotating. For ω_{ib}^{b} exhibiting pure coning motion (magnitude being constant but the vetor rotating), a fixed axis in the B frame that is approximately perpendicular to the plane of the rotating ω_{ib}^{b} vector will generate a conical surface as the angular rate motion ensues [6].*

Example 5. Integrate (140) for the case of the classical coning motion [2] where

$$\omega(t) = \begin{pmatrix} \theta\omega_c\cos(\omega_c t) \\ -\theta\omega_c\sin(\omega_c t) \\ 0 \end{pmatrix} \qquad (155)$$

The theoretically predicted value for ϕ is

$$\phi = \phi_0 + \begin{pmatrix} \theta\sin(\omega_c t) \\ \theta(\cos(\omega_c t) - 1) \\ \frac{1}{2}\theta^2\omega_c t \end{pmatrix} \tag{156}$$

where ϕ_0 is the initial condition. Specifically, if $\theta = 10^{-3}$ radians, $\omega_c = 20\pi$ rad / sec, the result of $\phi_z(10)$ is

$$\phi_z(10) = \frac{1}{2} \times (10^{-3})^2 \times 20\pi \times 10 = 10^{-4}\pi \text{ radians} \tag{157}$$

Example 6. Given the time history of the orientation of a rigid body [1], what was the angular velocity that generated that specified time history? For example, suppose the time history

$$\phi = \begin{pmatrix} \theta\sin(\omega_c t) \\ \theta\cos(\omega_c t) \\ 0 \end{pmatrix} \tag{158}$$

is given. Particularly here $\phi_0 = \phi(t=0) = \begin{pmatrix} 0 \\ \theta \\ 0 \end{pmatrix}$, this is the classical coning motion where $\|\phi\| = \theta$. using the equation in [2]

$$\omega = \dot{\phi} - \frac{1-\cos(\|\phi\|)}{\|\phi\|^2}\phi \times \dot{\phi} + \frac{1}{\|\phi\|^2}\left(1 - \frac{\sin(\|\phi\|)}{\|\phi\|}\right)\phi \times (\phi \times \dot{\phi}) \tag{159}$$

it gives

$$\begin{aligned} \omega &= \begin{pmatrix} \omega_c\theta\cos(\omega_c t) \\ -\omega_c\theta\sin(\omega_c t) \\ 0 \end{pmatrix} - \frac{1-\cos(\theta)}{\theta^2}\begin{pmatrix} 0 \\ 0 \\ -\omega_c\theta^2 \end{pmatrix} + \frac{1}{\theta^2}\left(1 - \frac{\sin(\theta)}{\theta}\right)\begin{pmatrix} -\omega_c\theta^3\cos(\omega_c t) \\ \omega_c\theta^3\sin(\omega_c t) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \omega_c\theta\cos(\omega_c t) \\ -\omega_c\theta\sin(\omega_c t) \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -\omega_c(1-\cos(\theta)) \end{pmatrix} + (\theta - \sin(\theta))\begin{pmatrix} -\omega_c\cos(\omega_c t) \\ \omega_c\sin(\omega_c t) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \omega_c\sin(\theta)\cos(\omega_c t) \\ -\omega_c\sin(\theta)\sin(\omega_c t) \\ \omega_c(1-\cos(\theta)) \end{pmatrix} \end{aligned} \tag{160}$$

where ω_c is called the coning frequency.

APPENDIX

Trigonometric functions

Trigonometric functions differentiation

$$\frac{d}{dx}(\sin(x)) = \cos(x), \quad \frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \frac{1}{\cos^2(x)} = \sec^2(x), \quad \frac{d}{dx}(\cot(x)) = -\frac{1}{\sin^2(x)} = -\csc^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \frac{\sin(x)}{\cos^2(x)} = \tan(x)\sec(x), \quad \frac{d}{dx}(\csc(x)) = -\frac{\cos(x)}{\sin^2(x)} = -\cot(x)\csc(x)$$

New proof of Bortz equation

QUESTION. *Derive Bortz equation [1]*

$$\frac{{}^B d\vec{\varphi}}{\mathrm{dt}} = {}^A\vec{\omega}^B + \frac{1}{2}\vec{\varphi} \times {}^A\vec{\omega}^B + \frac{1}{\varphi^2}\left(1 - \frac{\varphi\sin(\varphi)}{2(1-\cos(\varphi))}\right)\vec{\varphi} \times (\vec{\varphi} \times {}^A\vec{\omega}^B)$$

from the standard form of simple rotation dynamics in Chapter 9.1 [4]

$$\begin{aligned} {}^{A}\vec{\omega}^{B} &= \sin(\theta)\frac{{}^{B}d\vec{\lambda}}{\mathrm{dt}} + \dot{\theta}\vec{\lambda} + (\cos(\theta)-1)\vec{\lambda}\times\frac{{}^{B}d\vec{\lambda}}{\mathrm{dt}} \\ &= \sin(\theta)\frac{{}^{A}d\vec{\lambda}}{\mathrm{dt}} + \dot{\theta}\vec{\lambda} + (\cos(\theta)-1)\vec{\lambda}\times\frac{{}^{A}d\vec{\lambda}}{\mathrm{dt}} \end{aligned}$$

Proof.

$$\begin{aligned} {}^{A}\vec{\omega}^{B} &= \sin(\theta)\frac{{}^{B}d\vec{\lambda}}{\mathrm{dt}} + \dot{\theta}\vec{\lambda} + (\cos(\theta)-1)\vec{\lambda}\times\frac{{}^{B}d\vec{\lambda}}{\mathrm{dt}} && (161) \\ &= \sin(\theta)\frac{{}^{A}d\vec{\lambda}}{\mathrm{dt}} + \dot{\theta}\vec{\lambda} + (\cos(\theta)-1)\vec{\lambda}\times\frac{{}^{A}d\vec{\lambda}}{\mathrm{dt}} && (162) \end{aligned}$$

change varible $\varphi = \theta$, define $\vec{\varphi} = \varphi\vec{\lambda}$, and then

$$\begin{aligned} \frac{{}^{B}d\vec{\lambda}}{\mathrm{dt}} &= \frac{{}^{B}d}{\mathrm{dt}}\left(\frac{1}{\varphi}\cdot\vec{\varphi}\right) \\ &= \vec{\varphi}\cdot\frac{{}^{B}d}{\mathrm{dt}}\left(\frac{1}{\varphi}\right) + \frac{1}{\varphi}\cdot\frac{{}^{B}d}{\mathrm{dt}}(\vec{\varphi}) \\ &= -\frac{\dot{\varphi}\vec{\varphi}}{\varphi^2} + \frac{1}{\varphi}\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}} && (163) \end{aligned}$$

plug $\vec{\lambda} = \frac{\vec{\varphi}}{\varphi}$ and (163) into the (161),

$$\begin{aligned} {}^{A}\vec{\omega}^{B} &= \sin(\varphi)\left(-\frac{\dot{\varphi}\vec{\varphi}}{\varphi^2} + \frac{1}{\varphi}\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right) + \dot{\varphi}\left(\frac{\vec{\varphi}}{\varphi}\right) + (\cos(\varphi)-1)\left(\frac{\vec{\varphi}}{\varphi}\right)\times\left(-\frac{\dot{\varphi}\vec{\varphi}}{\varphi^2} + \frac{1}{\varphi}\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right) \\ &= \left(1-\frac{\sin(\varphi)}{\varphi}\right)\frac{\dot{\varphi}\vec{\varphi}}{\varphi} + \frac{\sin(\varphi)}{\varphi}\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}} + (\cos(\varphi)-1)\left(-\frac{\dot{\varphi}}{\varphi^3}(\vec{\varphi}\times\vec{\varphi}) + \frac{1}{\varphi^2}\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)\right) && (164) \end{aligned}$$

use

$$\begin{aligned} \vec{\varphi}\times\vec{\varphi} &= (\varphi\vec{\lambda})\times(\varphi\vec{\lambda}) \\ &= \varphi^2(\vec{\lambda}\times\vec{\lambda}) \\ &= 0 && (165) \\ \vec{\varphi}\cdot\vec{\varphi} &= (\varphi\vec{\lambda})\cdot(\varphi\vec{\lambda}) \\ &= \varphi^2(\vec{\lambda}\cdot\vec{\lambda}) \\ &= \varphi^2 && (166) \\ \vec{\varphi}\cdot\frac{{}^{B}d}{\mathrm{dt}}(\vec{\varphi}) &= (\varphi\vec{\lambda})\cdot\frac{{}^{B}d}{\mathrm{dt}}(\varphi\vec{\lambda}) \\ &= (\varphi\vec{\lambda})\cdot\left(\dot{\varphi}\vec{\lambda} + \varphi\frac{{}^{B}d\vec{\lambda}}{\mathrm{dt}}\right) \\ &= \varphi\dot{\varphi}(\vec{\lambda}\cdot\vec{\lambda}) + \varphi^2\left(\vec{\lambda}\cdot\frac{{}^{B}d\vec{\lambda}}{\mathrm{dt}}\right) \\ &= \varphi\dot{\varphi}(1) + \varphi^2(0) \\ &= \varphi\dot{\varphi} && (167) \end{aligned}$$

simplify (164) into

$$\begin{aligned} {}^{A}\vec{\omega}^{B} &= \left(1-\frac{\sin(\varphi)}{\varphi}\right)\frac{\dot{\varphi}\vec{\varphi}}{\varphi} + \frac{\sin(\varphi)}{\varphi}\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}} + (\cos(\varphi)-1)\left(-\frac{\dot{\varphi}}{\varphi^3}(0) + \frac{1}{\varphi^2}\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)\right) \\ &= \left(1-\frac{\sin(\varphi)}{\varphi}\right)\frac{\dot{\varphi}\vec{\varphi}}{\varphi} + \frac{\sin(\varphi)}{\varphi}\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}} + \frac{\cos(\varphi)-1}{\varphi^2}\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right) && (168) \end{aligned}$$

left cross multiply (168) by $\vec{\varphi}$,

$$\begin{aligned}\vec{\varphi}\times{}^{A}\vec{\omega}^{B} &= \left(1-\frac{\sin(\varphi)}{\varphi}\right)\frac{\dot{\varphi}(\vec{\varphi}\times\vec{\varphi})}{\varphi}+\frac{\sin(\varphi)}{\varphi}\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)+\frac{\cos(\varphi)-1}{\varphi^2}\vec{\varphi}\times\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)\\ &= \left(1-\frac{\sin(\varphi)}{\varphi}\right)\frac{\dot{\varphi}(0)}{\varphi}+\frac{\sin(\varphi)}{\varphi}\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)+\frac{\cos(\varphi)-1}{\varphi^2}\vec{\varphi}\times\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)\\ &= \frac{\sin(\varphi)}{\varphi}\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)+\frac{\cos(\varphi)-1}{\varphi^2}\vec{\varphi}\times\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)\end{aligned}\tag{169}$$

employ identity $a\times b\times c=b(a\cdot c)-c(a\cdot b)$,

$$\begin{aligned}\vec{\varphi}\times\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right) &= -\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\times\vec{\varphi}\\ &= -\left(\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}(\vec{\varphi}\cdot\vec{\varphi})-\vec{\varphi}\left(\vec{\varphi}\cdot\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)\right)\\ &= -\left(\varphi^2\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}-\varphi\dot{\varphi}\vec{\varphi}\right)\\ &= -\varphi^2\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\varphi\dot{\varphi}\vec{\varphi}\end{aligned}\tag{170}$$

we can simplify (169) into

$$\begin{aligned}\vec{\varphi}\times{}^{A}\vec{\omega}^{B} &= \frac{\sin(\varphi)}{\varphi}\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)+\frac{\cos(\varphi)-1}{\varphi^2}\left(-\varphi^2\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\varphi\dot{\varphi}\vec{\varphi}\right)\\ &= \frac{\sin(\varphi)}{\varphi}\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)+(1-\cos(\varphi))\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\frac{(\cos(\varphi)-1)\dot{\varphi}\vec{\varphi}}{\varphi}\end{aligned}\tag{171}$$

left cross multiply (171) by $\vec{\varphi}$,

$$\begin{aligned}\vec{\varphi}\times(\vec{\varphi}\times{}^{A}\vec{\omega}^{B}) &= \frac{\sin(\varphi)}{\varphi}\vec{\varphi}\times\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)+(1-\cos(\varphi))\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)+\frac{(\cos(\varphi)-1)\dot{\varphi}}{\varphi}(\vec{\varphi}\times\vec{\varphi})\\ &= \frac{\sin(\varphi)}{\varphi}\left(-\varphi^2\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\varphi\dot{\varphi}\vec{\varphi}\right)+(1-\cos(\varphi))\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)+\frac{(\cos(\varphi)-1)\dot{\varphi}}{\varphi}(0)\\ &= -\varphi\sin(\varphi)\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+(1-\cos(\varphi))\left(\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}\right)+\sin(\varphi)\dot{\varphi}\vec{\varphi}\end{aligned}\tag{172}$$

further it can be obtained

$$\begin{aligned}\frac{1}{\varphi^2}\left(1-\frac{\varphi\sin(\varphi)}{2(1-\cos(\varphi))}\right)\vec{\varphi}\times(\vec{\varphi}\times{}^{A}\vec{\omega}^{B}) &= -\frac{\sin(\varphi)}{\varphi}\left(1-\frac{\varphi\sin(\varphi)}{2(1-\cos(\varphi))}\right)\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\left(\frac{1-\cos(\varphi)}{\varphi^2}-\right.\\ &\quad\left.\frac{\sin(\varphi)}{2\varphi}\right)\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\left(\frac{\sin(\varphi)}{\varphi^2}-\frac{\sin^2(\varphi)}{2\varphi(1-\cos(\varphi))}\right)\dot{\varphi}\vec{\varphi}\\ &= \left(-\frac{\sin(\varphi)}{\varphi}+\frac{1+\cos(\varphi)}{2}\right)\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\left(\frac{1-\cos(\varphi)}{\varphi^2}-\right.\\ &\quad\left.\frac{\sin(\varphi)}{2\varphi}\right)\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\left(\frac{\sin(\varphi)}{\varphi}-\frac{1+\cos(\varphi)}{2}\right)\frac{\dot{\varphi}\vec{\varphi}}{\varphi}\end{aligned}\tag{173}$$

according to (168) and (171),

$$\begin{aligned}{}^{A}\vec{\omega}^{B}+\frac{1}{2}\vec{\varphi}\times{}^{A}\vec{\omega}^{B} &= \left(\frac{\sin(\varphi)}{\varphi}+\frac{1-\cos(\varphi)}{2}\right)\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\left(\frac{\cos(\varphi)-1}{\varphi^2}+\frac{\sin(\varphi)}{2\varphi}\right)\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\left(1-\right.\\ &\quad\left.\frac{\sin(\varphi)}{\varphi}+\frac{\cos(\varphi)-1}{2}\right)\frac{\dot{\varphi}\vec{\varphi}}{\varphi}\\ &= \left(\frac{\sin(\varphi)}{\varphi}+\frac{1-\cos(\varphi)}{2}\right)\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\left(\frac{\cos(\varphi)-1}{\varphi^2}+\frac{\sin(\varphi)}{2\varphi}\right)\vec{\varphi}\times\frac{{}^{B}d\vec{\varphi}}{\mathrm{dt}}+\\ &\quad\left(\frac{\cos(\varphi)+1}{2}-\frac{\sin(\varphi)}{\varphi}\right)\frac{\dot{\varphi}\vec{\varphi}}{\varphi}\end{aligned}\tag{174}$$

essentially the addition of (173) and (174) produces

$$ {}^A\vec{\omega}^B + \frac{1}{2}\vec{\varphi} \times {}^A\vec{\omega}^B + \frac{1}{\varphi^2}\left(1 - \frac{\varphi \sin(\varphi)}{2(1-\cos(\varphi))}\right)\vec{\varphi} \times (\vec{\varphi} \times {}^A\vec{\omega}^B) \;=\; \frac{{}^B d\vec{\varphi}}{\mathrm{dt}} \tag{175} $$

□

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