# The Derivation of Sample Variance's unbiasedness

An article in MTCF2e

Maiar

January 23, 2013

# Contents

	Question				
	1.1	Variance	1		
	1.2	Sample Variance	1		
	1.3	Unbiasedness	2		
2	Answer				
	2.1	Derivation	4		
	2.2	Tips	Ę		

## Chapter 1

## Question

#### 1.1 Variance

Well, and here begins my first article (actually a book, that's not a joke...) written in LATEX2e.

It's defined as follows:

$$p_i = P(X = x_i) \tag{1.1}$$

$$E(X) = \sum_{i=1}^{n} x_i p_i \tag{1.2}$$

$$D(X) = E\left(\left(X - E(X)\right)^{2}\right) = \sum_{i=1}^{n} \left(x_{i} - E(X)\right)^{2} p_{i}$$
 (1.3)

#### 1.2 Sample Variance

We introduce a new variable  $\overline{X}$  as equation (1.4) on page 1 shows:

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{1.4}$$

Then

$$E(\overline{X}) = \frac{1}{n} E\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \mu$$
 (1.5)

$$D(\overline{X}) = \frac{1}{n^2} D\left(\sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_{i=1}^n D(X_i) = \frac{\sigma^2}{n}$$
 (1.6)

<sup>1</sup> 这是我的第一个注脚。

2 Question

Sample Variance is defined as equation (1.7):

$$S^{2} = \frac{1}{n-1} \sum_{i=1}^{n} (X_{i} - \overline{X})^{2}$$
(1.7)

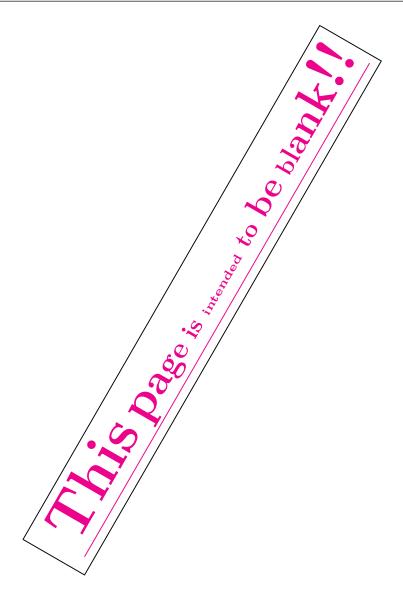
#### 1.3 Unbiasedness

The unbiasedness of Sample Mean and Sample Variance is as follows:

$$\begin{cases}
E(\overline{X}) = E(X) \\
E(S^2) = D(X)
\end{cases}$$
(1.8)

Now we get to solve it.

1.3 Unbiasedness 3



## Chapter 2

### Answer

#### 2.1 Derivation

Firstly, we all know:

$$D(X) = E(X^{2}) - (E(X))^{2}$$
$$E(X^{2}) = D(X) + (E(X))^{2}$$

And the means and variances:

$$E(X_i) = E(\overline{X}) = \mu$$
$$D(X_i) = \sigma^2, D(\overline{X}) = \frac{\sigma^2}{n}$$

So we get

$$E(X_{i}^{2}) = D(X_{i}) + \mu^{2} = \sigma^{2} + \mu^{2}$$

$$E(\overline{X}^{2}) = D(\overline{X}) + \mu^{2} = \frac{\sigma^{2}}{n} + \mu^{2}$$

$$E(\overline{X}X_{i}) = E\left(\frac{X_{1}X_{i} + X_{2}X_{i} + \dots + X_{i}^{2} + \dots + X_{n}X_{i}}{n}\right)$$

$$= \frac{1}{n}\left(\sum_{\substack{k=0\\k\neq i}}^{n} E(X_{k}X_{i}) + E(X_{i}^{2})\right)$$

$$= \frac{1}{n}((n-1)\mu^{2} + (\sigma^{2} + \mu^{2}))$$

$$= \frac{\sigma^{2}}{n} + \mu^{2}$$

$$= E(\overline{X}^{2})$$

2.2 Tips 5

Now let's begin the derivation:

$$E(S^{2}) = \frac{1}{n-1} E\left(\sum_{i=1}^{n} (X_{i} - \overline{X})^{2}\right)$$

$$= \frac{1}{n-1} E\left(\sum_{i=1}^{n} \left(X_{i}^{2} - 2\overline{X}X_{i} + \overline{X}^{2}\right)\right)$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} \left(E\left(X_{i}^{2} - 2\overline{X}X_{i} + \overline{X}^{2}\right)\right)$$

$$= \frac{1}{n-1} \left(2n\mu^{2} + (n+1)\sigma^{2} - 2\sum_{i=1}^{n} \left(E(\overline{X}X_{i})\right)\right)$$

$$= \frac{1}{n-1} \left(2n\mu^{2} + (n+1)\sigma^{2} - 2n(\frac{\sigma^{2}}{n} + \mu^{2})\right)$$

$$= \sigma^{2}$$

$$= D(X)$$

That's all. Hope you enjoy it.

#### 2.2 Tips

What's the difference between  $x_i$  and  $X_i$ ??

to click the blank page 3 and comment on my article ... and here it ends.